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Algebraic structures on modules of diagrams

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ABSTRACT

There exists a graded algebra Λ acting in a natural way on many modules of 3-valent diagrams. Every simple Lie superalgebra with a nonsingular invariant bilinear form induces a character on Λ. The classical and exceptional Lie algebras and the Lie superalgebra $D(2,1,\alpha)$ produce eight distinct characters on Λ and eight distinct families of weight functions on chord diagrams. As a consequence we prove that weight functions coming from semisimple Lie superalgebras do not detect every element in the module $\mathcal{A}$ of chord diagrams. A precise description of Λ is conjectured.

0. Introduction

Vassiliev [22] has recently defined a new family of knot invariants. Actually every knot invariant with values in an abelian group may be seen as a linear map from the free $\mathbf{Z}$-module $\mathbf{Z}[\mathcal{K}]$ generated by isomorphism classes of knots. This module is a Hopf algebra and has a natural filtration $\mathbf{Z}[\mathcal{K}] = I_0 \supset I_1 \supset \cdots$ defined in terms of singular knots, and a Vassiliev invariant of order n is an invariant which is trivial on I_{n+1}. The coefficients of Jones [11], Freyd et al. [9], Kauffman [13] polynomials are Vassiliev invariants.

The associated graded Hopf algebra $\mathrm{Gr}\,\mathbf{Z}[\mathcal{K}] = \oplus_n I_n/I_{n+1}$ is finitely generated over $\mathbf{Z}$ in each degree, but its rank is completely unknown. Actually $\mathrm{Gr}\,\mathbf{Z}[\mathcal{K}]$ is a certain quotient of the graded Hopf algebra $\mathcal{A}$ of chord diagrams [1]. Every Vassiliev invariants of order n induces a weight function of degree n, (i.e. a linear form of degree n on $\mathcal{A}$). Conversely every weight function can be integrated (via the Kontsevich integral) to a knot invariant. Very few things are known about the algebra $\mathcal{A}$. Rationally, $\mathcal{A}$ is the symmetric algebra on a graded module $\mathcal{P}$, and the so-called Adams operations split $\mathcal{A}$ and $\mathcal{P}$ into a direct sum of modules defined in terms of unitrivalent diagrams. The rank of $\mathcal{P}$ is known in degrees < 10.

Every Lie algebra equipped with a nonsingular invariant bilinear form and a finite-dimensional representation induces a weight function on $\mathcal{A}$. It was conjectured in [1] that the weight functions corresponding to the classical simple Lie algebras detect every nontrivial element in $\mathcal{A}$.

In this paper,[1] we define a graded algebra Λ acting on many modules of diagrams like $\mathcal{P}$. Moreover we construct for every Lie algebra equipped with a nonsingular invariant bilinear form, a linear form on these modules and a character on Λ. With this procedure, we construct eight characters from Λ to polynomial algebras of one or two variables. These eight characters are algebraically independent. As a consequence, we construct a primitive element in $\mathcal{A}$ which is rationally nontrivial and killed by all semisimple Lie algebras and Lie superalgebras equipped with a nonsingular invariant bilinear form and a finite-dimensional representation.

In the first section several families of modules of diagrams are defined.

In Section 2, we construct a transformation t of degree 1 acting on some of these modules.

In Section 3, we construct the algebra Λ. This algebra contains the element t.

In Section 4, some modules of diagrams are completely described in terms of Λ.

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[1] This is an expanded and updated version of a 1995 preprint.

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In Section 5, we define many elements in Λ and construct a graded algebra homomorphism from R_0 to Λ, where R_0 is a subalgebra of a polynomial algebra R with three variables of degree 1, 2 and 3.

In Section 6, we construct many weight functions and show that every simple quadratic Lie superalgebra induces a well-defined character on Λ

In Section 7, we construct the eight characters.

Using these characters, many results on Λ are proven in the last section. In particular, the morphism $R_0 \longrightarrow \Lambda$ factors through a quotient R_0/I where I is an ideal in R generated by a polynomial $P \in R_0$ of degree 16 and the induced morphism $R_0/I \longrightarrow \Lambda$ is conjectured to be an isomorphism.

1. Modules of diagrams

By a 3-valent graph we mean a graph where every vertex is 1-valent or 3-valent. A 3-valent graph is defined by local conditions. So in such a graph an edge may be a loop and two distinct edges may have common boundary points. The set of 1-valent vertices of a 3-valent graph K will be called its boundary and denoted by ∂K.

Let Γ be a curve, i.e. a compact 1-dimensional manifold and X be a finite set. A (Γ, X)-*diagram* is a finite 3-valent graph D equipped with the following data:

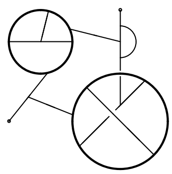
— an isomorphism from the disjoint union of Γ and X to a subgraph of D sending $\partial\Gamma \cup X$ bijectively to ∂D

— for every 3-valent vertex x of D, a cyclic ordering of the set of oriented edges ending at x.

The class of (Γ, X)-diagrams will be denoted by $\mathcal{D}(\Gamma, X)$.

Usually, a (Γ, X)-diagram will be represented by a 3-valent graph immersed in the plane in such a way that, at every 3-valent vertex, the cyclic ordering is given by the orientation of the plane.

Example of a (Γ, X)-diagram where Γ has two closed components and X has two elements:

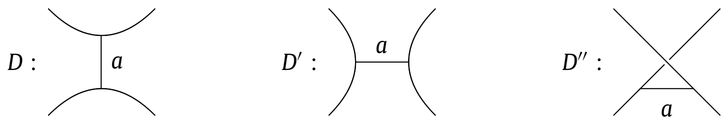


Let $\mathcal{C}$ be a subclass of $\mathcal{D}(\Gamma, X)$ which is closed under arbitrary changes of the cyclic orderings. Let k be a commutative ring. Denote by $\mathcal{A}_k(\mathcal{C})$ the quotient of the free k-module generated by the isomorphism classes of (Γ, X)-diagrams in $\mathcal{C}$ by the following relations:

— if D is a (Γ, X)-diagram in $\mathcal{C}$, and D' is obtained from K by changing the cyclic ordering at one vertex, we have

(AS) $\quad D' \equiv -D$

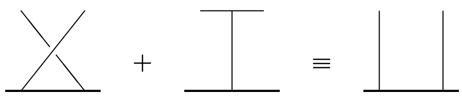
— if D, D', D'' are three (Γ, X)-diagrams in $\mathcal{C}$ which differ only near an edge a in the following way:



we have

(IHX) $\quad D \equiv D' - D''$.

Remark. If the edge meets the curve Γ the relation (IHX) is called (STU) in [1]:



The module $\mathcal{A}_k(\mathcal{C})$ is a graded k-module. The degree $\partial^\circ D$ of a (Γ, X)-diagram D is $-\chi(D)$ where χ is the Euler characteristic.

By considering different classes of diagrams, we get the following examples of graded modules:

— the module $\mathcal{A}_k(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}'(\Gamma, X)$ of (Γ, X)-diagrams D such that every connected component of D meets Γ or X

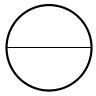
— the module $\mathcal{A}_k^c(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}^c(\Gamma, X)$ of (Γ, X)-diagrams D such that $D \setminus \Gamma$ is connected and nonempty (*connected case*)

— the module $\mathcal{A}_k^s(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}^s(\Gamma, X)$ of (Γ, X)-diagrams D such that $D \setminus \Gamma$ is connected and has at least one 3-valent vertex (*special case*)

— the module $\mathcal{A}_k(\Gamma) = \mathcal{A}_k(\Gamma, \emptyset)$
— the module $\mathcal{A}^c_k(\Gamma) = \mathcal{A}^c_k(\Gamma, \emptyset)$
— the module $F_k(X) = \mathcal{A}^c_k(\emptyset, X)$. If X is the set $[n] = \{1, \ldots, n\}$, the module $F_k(X)$ will be denoted by $F_k(n)$
— the module ${}_X\Delta_{kY}$, where X and Y are finite sets and $\mathcal{C}$ is the class of all $(\emptyset, X \coprod Y)$-diagrams.

The most interesting case is $k = \mathbf{Q}$. So the modules $\mathcal{A}_{\mathbf{Q}}(\mathcal{C})$, $\mathcal{A}_{\mathbf{Q}}(\Gamma, X)$, $\mathcal{A}^c_{\mathbf{Q}}(\Gamma, X)$, $\mathcal{A}^s_{\mathbf{Q}}(\Gamma, X) \ldots$ will be simply denoted by $\mathcal{A}(\mathcal{C})$, $\mathcal{A}(\Gamma, X)$, $\mathcal{A}^c(\Gamma, X)$, $\mathcal{A}^s(\Gamma, X) \ldots$.

The module $\mathcal{A}_k(\Gamma)$ is closely related to the theory of links. In the case of knots, the Kontsevich integral provides a universal Vassiliev invariant with values in a completion of the quotient of the module $\mathcal{A} = \mathcal{A}_{\mathbf{Q}}(S^1) = \mathcal{A}(S^1)$ by some submodule I [1]. The module $\mathcal{A}$ is actually a commutative and cocommutative Hopf algebra (the product corresponds to the connected sum of knots) and I is the ideal generated by the following diagram of degree 1:



Remark. The definition of the module $\mathcal{A}_k(\Gamma)$ is slightly different from the classical one. The classical definition needs an orientation of Γ, but cyclic orderings at vertices in Γ are not part of the data. The relationship between these two definitions come from the fact that, if Γ is oriented, there is a canonical choice for the cyclic ordering of edges ending at each vertex in Γ.

Let $\mathcal{P}_k = \mathcal{A}^c_k(S^1)$ and $\mathcal{A}_k = \mathcal{A}_k(S^1)$. The inclusion $\mathcal{D}^c(S^1, \emptyset) \subset \mathcal{D}(S^1, \emptyset)$ induces a linear map from $\mathcal{P}_k$ to $\mathcal{A}_k$ and a morphism of Hopf algebras from $S(\mathcal{P}_k)$ to $\mathcal{A}_k$.

Proposition 1.1. *The morphism $S(\mathcal{P}_{\mathbf{Z}}) \to \mathcal{A}_{\mathbf{Z}}$ is surjective with finite kernel in each degree.*

Proof. For $n > 0$, denote by $\mathcal{E}_n$ the submodule of $\mathcal{A}_{\mathbf{Z}}$ generated by the diagrams D such that $D \setminus S^1$ has at most n components. Because of relation STU, it is easy to see that, mod $\mathcal{E}_n$, $\mathcal{E}_{n+1}$ is generated by connected sums $K_1 \sharp K_2 \cdots \sharp K_{n+1}$ where $K_i \setminus S^1$ are connected. That proves, by induction, that the canonical map from $S(\mathcal{P}_{\mathbf{Z}})$ to $\mathcal{A}_{\mathbf{Z}}$ is surjective. Because $S(\mathcal{P}_{\mathbf{Z}})$ and $\mathcal{A}_{\mathbf{Z}}$ are finitely generated over $\mathbf{Z}$ in each degree, it is enough to prove that the map from $S(\mathcal{P}_{\mathbf{Z}})$ to $\mathcal{A}_{\mathbf{Z}}$ is a rational isomorphism, and because $S(\mathcal{P}_{\mathbf{Z}})$ and $\mathcal{A}_{\mathbf{Z}}$ are commutative and cocommutative Hopf algebras, it is enough to prove that the map from $\mathcal{P} = \mathcal{P}_{\mathbf{Q}}$ to $\mathcal{A} = \mathcal{A}_{\mathbf{Q}}$ is an isomorphism from $\mathcal{P}$ to the module of primitives of $\mathcal{A}$.

Consider the module $\mathcal{C}_p$ of 3-valent diagrams with p univalent vertices and the module $\mathcal{C}^c_p$ of connected 3-valent diagrams with p univalent vertices. In [1] Bar-Natan constructs a rational isomorphism from $\mathcal{A}$ to the direct sum $\oplus_{p>0}\mathcal{C}_p$ that respects the comultiplication. In the same way we have a rational isomorphism from $\mathcal{P}$ to $\oplus_{p>0}\mathcal{C}^c_p$.

Therefore $\mathcal{P}$ is isomorphic to the module of primitives of $\mathcal{A}$. $\square$

Very little is known about $\mathcal{A}$ and $\mathcal{P}$. They are finitely generated modules in each degree. Their ranks are known in degrees ≤ 9. For $\mathcal{P}$, the ranks are: 1, 1, 1, 2, 3, 5, 8, 12, 18 [1]. Some linear forms (called weight functions) on $\mathcal{A}$ (coming from Lie algebras) are known. Rationally the module $\mathcal{P}$ splits into a direct sum of modules of connected 3-valent diagrams $\mathcal{C}^c_n$ [1]. Actually the module $\mathcal{C}^c_n$ is defined in the same way as $F(n) = F_{\mathbf{Q}}(n)$ except that the bijection from $[n]$ to the set of 1-valent vertices is forgotten. Hence this splitting may be written in the following manner:

Proposition 1.2. *There is an isomorphism:*

$$\bigoplus_{n>0} \mathrm{H}_0(\mathfrak{S}_n, F(n)) \xrightarrow{\sim} \mathcal{P}.$$

The last module ${}_X\Delta_{kY}$ defined above will be used later. Actually these modules define a k-linear monoidal category Δ_k. The objects of Δ_k are finite sets, and the set of morphisms $\mathrm{Hom}(X, Y)$ is the module ${}_Y\Delta_{kX}$. The composition from ${}_X\Delta_{kY} \otimes {}_Y\Delta_{kZ}$ to ${}_X\Delta_{kZ}$ is obtained by gluing. In particular, for every finite set X, ${}_X\Delta_{kX}$ is a k-algebra.

The monoidal structure is given by the disjoint union of finite sets or diagrams.

For technical reasons we will use a modified degree for modules $F_k(X)$ and ${}_X\Delta_{kY}$:

— the degree of an element $u \in F_k(X)$ represented by a diagram D is $1 - \chi(D)$. So the degree of a tree is zero.

— the degree of an element $u \in {}_Y\Delta_{kX}$ represented by a diagram D is $-\chi(D, X)$. This degree is compatible with the structure of k-linear monoidal category.

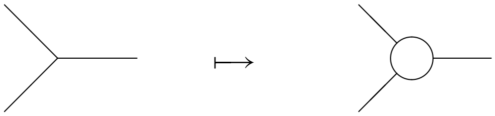
2. The transformation *t*

Let Γ be a curve and X be a finite set. We have three graded modules $\mathcal{A}_k(\Gamma, X)$, $\mathcal{A}^c_k(\Gamma, X)$ and $\mathcal{A}^s_k(\Gamma, X)$ and two canonical maps:

$$\mathcal{A}^s_k(\Gamma, X) \longrightarrow \mathcal{A}^c_k(\Gamma, X) \longrightarrow A_k(\Gamma, X).$$

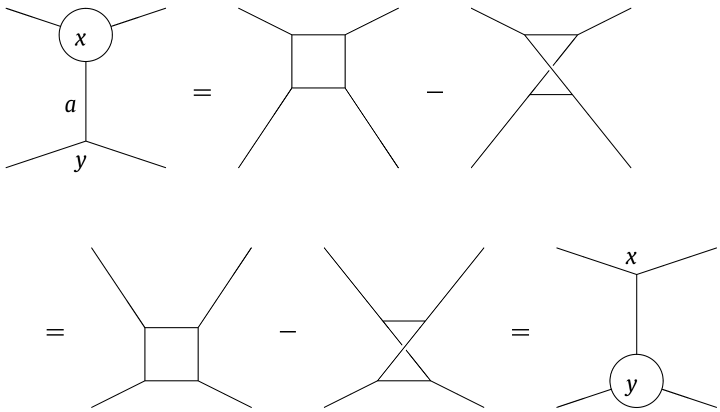
The second map is far to be surjective but the first one is an isomorphism except maybe in small degrees.

Let D be a (Γ, X)-diagram in the class $\mathcal{D}_k^s(\Gamma, X)$. Take a 3-valent vertex outside of Γ. Then it is possible to modify D near this vertex in the following way:



Theorem 2.1. *This transformation induces a well-defined endomorphism t of the module $\mathcal{A}_k^s(\Gamma, X)$.*

Proof. Let D be a diagram in the class $\mathcal{D}_k^s(\Gamma, X)$. Let a be an edge of D disjoint from the curve Γ. Denote vertices of a by x and y. Relations IHX imply the following:



Then transformations of D at x and y produce the same element in the module $\mathcal{A}_k^s(\Gamma, X)$. Since the complement of Γ in a diagram in $\mathcal{D}_s(\Gamma, X)$ is connected, the transformation t is well defined from the class $\mathcal{D}_k^s(\Gamma, X)$ to $\mathcal{A}_k^s(\Gamma, X)$.

It is easy to see that t is compatible with the AS relation. Consider an IHX relation

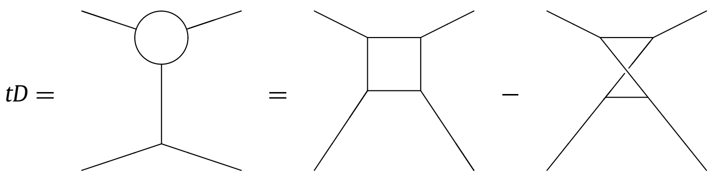
$$D \equiv D' - D'',$$

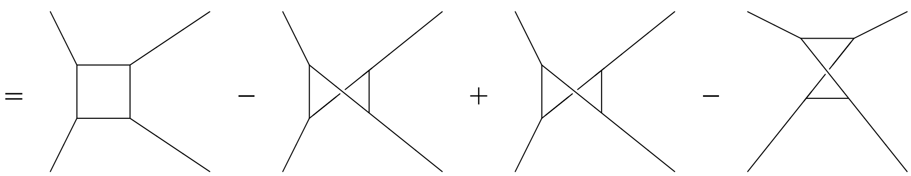
where D, D' and D'' differ only near an edge a. If there is a 3-valent vertex in D which is not in a and not in the curve Γ, it is possible to define tD, tD', and tD'' by using this vertex, and the relation

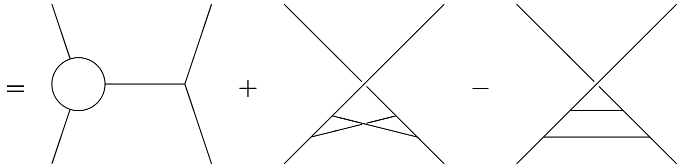
$$tD \equiv tD' - tD''$$

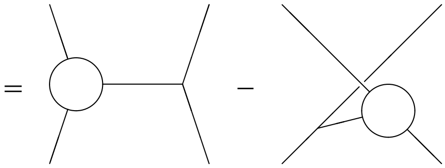
becomes obvious.

Suppose now $\Gamma \cup X \cup a$ contains every vertex in D. Then the edge a is not contained in Γ, and that is true also for D' and D''. Therefore a does not meet Γ, and we have:







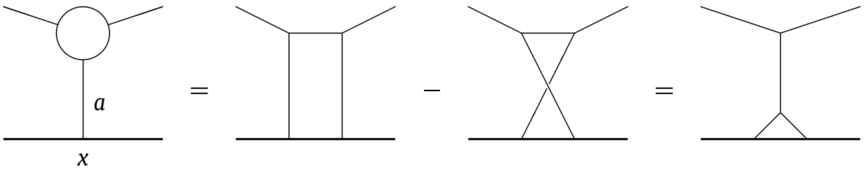


$= tD' - tD''$ □

Proposition 2.2. *If Γ is nonempty, the transformation t extends in a natural way to the module $\mathcal{A}_k^c(\Gamma, X)$.*

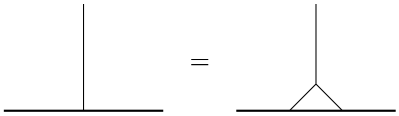
Proof. Let D be a diagram in the class $\mathcal{D}_k^c(\Gamma, X)$. Let x be a 3-valent vertex of D contained in Γ. This vertex in contained in an edge a in $D \setminus \Gamma$. If the diagram D lies in the class $\mathcal{D}_k^s(\Gamma, X)$, D has a vertex which is not in Γ. Therefore a has a vertex outside of Γ and we have

$tD =$



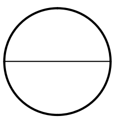
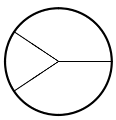
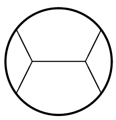
Hence t extends to the module $\mathcal{A}_k^c(\Gamma, X)$ by setting

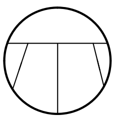
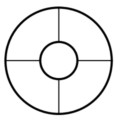
t



□

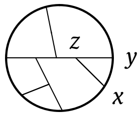
Example. The module $\mathcal{P}_k = \mathcal{A}_k^c(S^1) = \mathcal{A}_k^c(S^1, \emptyset)$ which is the module of primitives of the algebra of diagrams $\mathcal{A}_k$, has in degree ≤ 4 the following basis:

 $= \alpha$,  $= t\alpha$,  $= t^2\alpha$,

 $= t^3\alpha$ and 

Corollary. *Let D be a planar $(S^1, \emptyset)$-diagram of degree n such that the complement of S^1 in D is a tree. Then the class of D in the module $\mathcal{A}_k^c(S^1)$ is exactly $t^{n-1}\alpha$.*

Proof. The conditions satisfied by D imply that D contains a triangle xyz with an edge xy in the circle. By taking off the edge xz, we get a new diagram D' such that the complement of the circle in D' is still a planar tree. By induction, the class of D' in $\mathcal{A}_k^c(S^1)$ is $t^{n-2}\alpha$ and the result follows.



□

3. The algebra Λ

In this section we construct an algebra of diagrams acting on many modules of diagrams. In particular this algebra acts in a natural way on the modules $\mathcal{A}_k^s(\Gamma, X)$. Actually the element t is a particular element of Λ of degree 1.

The module $F_k(X)$ is equipped with an action of the symmetric group $\mathfrak{S}(X)$. But we can also define natural maps from $F_k(X)$ to $F_k(Y)$ in the following way:

Let D be a $(\emptyset, X \coprod Y)$-diagram such that every connected component of D meets X and Y. Then the gluing map along X induces a graded linear map φ_D from $F_k(X)$ to $F_k(Y)$. Actually the class $\mathcal{C}$ of $(\emptyset, X \coprod Y)$-diagrams satisfying this property induces a graded module ${}_X\Delta_{kY}^c = \mathcal{A}_k(\mathcal{C})$ and these modules give rise to a monoidal subcategory Δ_k^c of the category Δ_k. For every finite set X and Y the gluing map is a map from $F_k(X) \otimes {}_X\Delta_{kY}^c$ to $F_k(Y)$.

In particular we have two maps φ and φ' from $F_k(3)$ to $F_k(4)$ induced by the following diagrams:

Definition 3.1. Λ_k is the set of elements $u \in F_k(3)$ satisfying the following conditions:

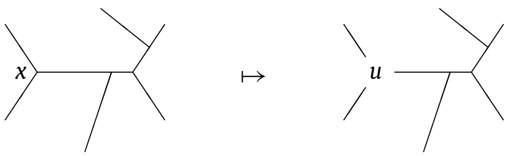
$$\varphi(u) = \varphi'(u)$$
$$\forall \sigma \in \mathfrak{S}_3, \qquad \sigma(u) = \varepsilon(\sigma)u$$

where ε is the signature homomorphism.

The module $\Lambda_{\mathbf{Q}}$ will be denoted by Λ.

Proposition 3.2. *The module Λ_k is a graded k-algebra acting on each module $\mathcal{A}_k^s(\Gamma, X)$.*

Proof. Let Γ be a curve and X be a finite set. Let D be a (Γ, X)-diagram such that $D \setminus \Gamma$ is connected and has some 3-valent vertex x. If u is an element of Λ_k, we can insert u in D near x and we get a linear combination of diagrams and therefore an element uD in $\mathcal{A}_k^s(\Gamma, X)$.



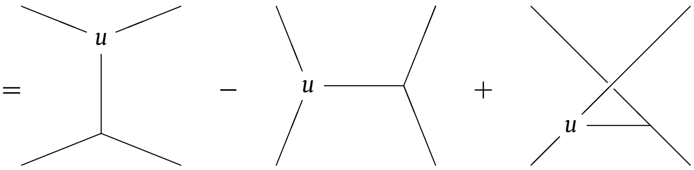
Since u is completely antisymmetric with respect to the $\mathfrak{S}_3$-action, uD does not depend on the given bijection from $[3] = \{1, 2, 3\}$ to the set of edges ending at x, but only on the cyclic ordering. Moreover, if this cyclic ordering is changed, uK is multiplied by -1. The first condition satisfied by u implies that the elements uK constructed by two consecutive vertices are the same. Since the complement of Γ in D is connected, uD does not depend on the choice of the vertex x, and uD is well defined.

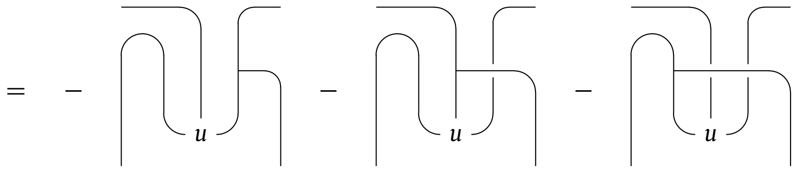
By construction, the rule $u \mapsto uD$ is a linear map from Λ_k to $\mathcal{A}_k^s(\Gamma, X)$ of degree $\partial^\circ D$. Since the transformation $D \mapsto uD$ is compatible with the AS relations, the only thing to check is to prove that this transformation is compatible with the IHX relations.

Consider an IHX relation $D \equiv D' - D''$ corresponding to an edge a in D. If D has a 3-valent vertex outside of a and Γ, it is possible to make the transformation ? $\mapsto$ u? by using a vertex which is not in a, and we get the equality: $uD = uD' - uD''$.

Otherwise a is outside of Γ and we have:

$uD - uD' + uD'' =$



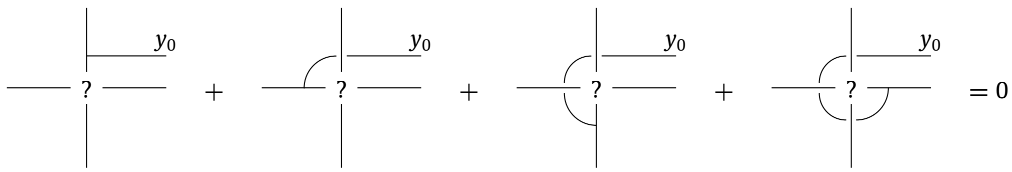


This last expression is trivial, because of Lemma 3.3 and the formula $uD = uD' - uD''$ is always true.

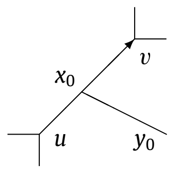
Therefore the transformation $? \mapsto u?$ is compatible with the IHX relation and induces a well-defined transformation from $\mathcal{A}_k^s(\Gamma, X)$ to itself. In particular, Λ_k acts on itself. Therefore this module is a k-algebra and $\mathcal{A}_k^s(\Gamma, X)$ is a Λ_k-module. □

Lemma 3.3. *Let X be a finite set and Y be the set X with one extra point y_0 added. Let D be a connected $(\emptyset, X)$-diagram. For every $x \in X$ denote by D_x the $(\emptyset, Y)$-diagram obtained by adding to D an extra edge from y_0 to a point in D near x, the cyclic ordering near the new vertex being given by taking the edge ending at y_0 first, the edge ending at x after and the last edge at the end.*

Then the element $\Sigma_x D_x$ is trivial in the module $F(Y)$.

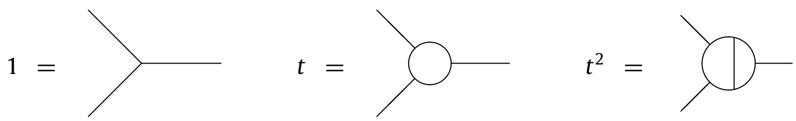


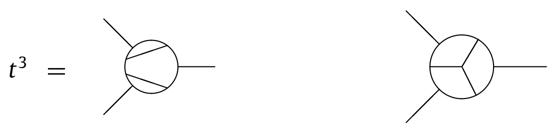
Proof. For every oriented edge a in D from a vertex u to a vertex v, we can connect y_0 to K by adding an extra edge from y_0 to a new vertex x_0 in a and we get a $(\emptyset, Y)$-diagram D_a where the cyclic ordering between edges ending at x_0 is (x_0u, x_0y_0, x_0v).



It is clear that the expression $D_a + D_b$ is trivial if b is the edge a with the opposite orientation. Moreover if a, b and c are the three edges starting from a 3-valent vertex of K, the sum $D_a + D_b + D_c$ is also trivial. Therefore the sum ΣD_a for all oriented edge a in D is trivial and is equal to the sum ΣD_a for all oriented edge a starting from a vertex in X. That proves the lemma. □

In degree less to 4, the module Λ_k is freely generated by the following diagrams:





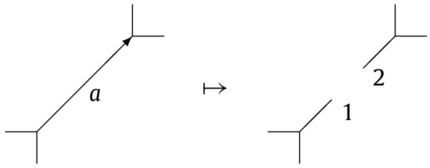
4. Structure of modules $F(n)$ for small values of n

The module $F_k(n)$ is a Λ_k-module except for $n = 0, 2$. But the submodule $F'_k(n) = \mathcal{A}^s_k(\emptyset, [n])$ of $F_k(n)$ generated by diagrams having at least one 3-valent vertex is a Λ_k-module. For $n \neq 0, 2$, $F'_k(n)$ is equal to $F_k(n)$ and for $n = 0, 2$, $F_k(n)$ is isomorphic to $k \oplus F'_k(n)$.

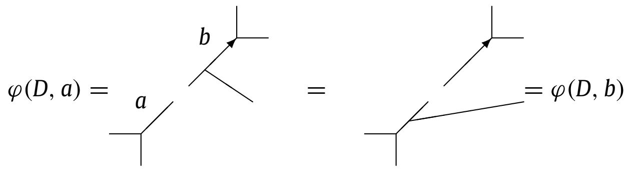
Proposition 4.1. *Connecting the elements of* $[2]$ *by an edge induces an isomorphism from* $F_k(2)$ *to* $F_k(0)$.

Proof. This map is clearly surjective.

Let D be a connected $(\emptyset, [0])$-diagram. Let a be an oriented edge of D. We can cut off a part of a and we get a $(\emptyset, [2])$-diagram $\varphi(D, a)$.



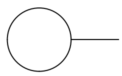
Let a and b be consecutive edges in D. Because of Lemma 3.3, we have:



Therefore $\varphi(K, a)$ is independent of the choice of a and induces a well-defined map from $F_k(0)$ to $F_k(2)$ which is obviously the inverse of the map above. □

Corollary 4.2. *The action of the symmetric group* $\mathfrak{S}_2$ *on* $F_k(2)$ *is trivial.*

Proposition 4.3. *The module* $F_k(1)$ *is isomorphic to* $k/2$ *and generated by the following diagram:*



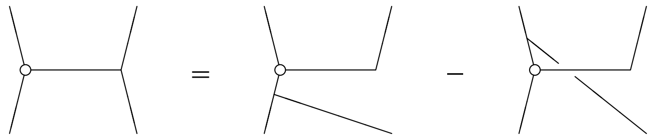
Proof. The diagram above is clearly a generator of $F_k(1)$ in degree 1, and the antisymmetric relation implies that this element is of order 2. Let D be a $(\emptyset, [1])$-diagram of degree > 1. We have:

and this last diagram contains the following diagram:

Proposition 4.4. *The quotient map from* $[3]$ *to a point induces a surjective map from* $F_k(3) \otimes_{\mathfrak{S}_3} k^-$ *to* $\mathrm{F}'(0)$ *and its kernel is a* $k/2$*-module.*

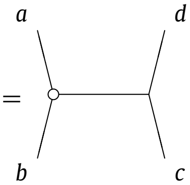
Proof. Here the group $\mathfrak{S}_3$ acts on $k = k^-$ via the signature. Actually, the module $F_k(3) \otimes_{\mathfrak{S}_3} k^-$ is isomorphic to the module $\mathcal{M}$ generated by connected 3-valent diagrams without univalent vertex, pointed by a vertex and equipped with a cyclic ordering near every vertex and where the relations are the AS relation everywhere and the IHS relation outside of the special vertex.

Because of Lemma 3.3, we have in $\mathcal{M}$:



Actually we have for every $n \geq 0$ a module $\widetilde{\mathrm{F}}(n)$ generated by connected diagrams K with $\partial K = [n]$ and pointed by a 3-valent vertex. The relations are the antisymmetric relation AS everywhere and the relation IHX outside of the special vertex and the relation above.

If $\{a, b, c, d\} = [4]$, we can set:

$\varphi(a, b, c, d) =$ 

This diagram belongs to $\widetilde{\mathrm{F}}(4)$ and is antisymmetric with respect to the transpositions $a \leftrightarrow b$ and $c \leftrightarrow d$. Let k^- be the maximal exterior power of the module generated by the elements of [4]. Define the element $\psi(a, b, c, d)$ in $k^- \otimes \widetilde{\mathrm{F}}(4)$ by: $\psi(a, b, c, d) = a{\wedge}b{\wedge}c{\wedge}d \otimes \varphi(a, b, c, d)$. By construction $\psi(a, b, c, d)$ depends only on the subset $\{c, d\}$ of [4]. So we set: $\psi(a, b, c, d) = f(c, d)$.

The relation obtained by Lemma 3.3 is:

$$\sum_{x \neq a} f(a, x) = 0$$

for every a in [4].

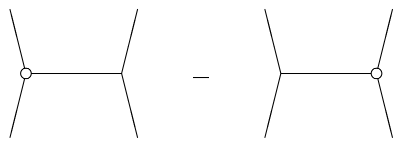
For $\{a, b, c, d\} = [4]$, set: $g(a, b) = f(a, b) - f(c, d)$. We have:

$$f(a, b) + f(a, c) + f(a, d) = 0 = f(b, a) + f(b, c) + f(b, d)$$
$$\Longrightarrow g(a, c) = g(b, c).$$

Then $u = g(a, b)$ does not depend on $\{a, b\}$ and we have:

$$u = g(a, b) = f(a, b) - f(c, d) = -g(c, d) = -u.$$

Therefore the diagram



is killed by 2 and invariant under the action of $\mathfrak{S}_4$.

Let α be an element in $F'_k(0)$ represented by a 3-valent diagram D. Take a vertex x_0 in D. The pair (K, x_0) represents a well-defined element β in the module $\mathcal{M} \simeq F_k(3) \otimes_{\mathfrak{S}_3} k^-$ and 2β does not depend on the choice of the vertex x_0. Hence the rule $\alpha \mapsto 2\beta$ is a well-defined map λ from $F'_k(0)$ to $F_k(3) \otimes_{\mathfrak{S}_3} k^-$. Denote by μ the canonical map from $F_k(3) \otimes_{\mathfrak{S}_3} k^-$ to $F'(0)$. We have:

$$\mu\lambda = 2 \quad \text{and} \quad \lambda\mu = 2$$

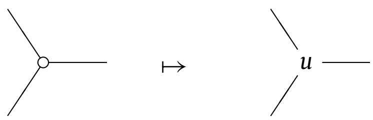
and Proposition 4.4 follows. $\square$

Proposition 4.5. *Let $F_k(3)^-$ be the submodule of $F_k(3)$ defined by:*

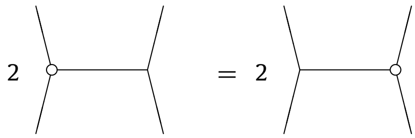
$$\forall u \in F_k(3), \qquad u \in F_k(3)^- \Leftrightarrow (\forall \sigma \in \mathfrak{S}_3, \sigma(u) = \varepsilon(\sigma)u)$$

where ε is the signature homomorphism. Then Λ_k is a submodule of $F_k(3)^-$ and the quotient $F_k(3)^-/\Lambda_k$ is a $k/2$-module.

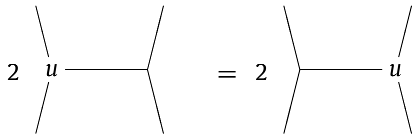
Proof. Let u be an element of $F_k(3)^-$. If v is an element of $\widetilde{F}_k(4)$ represented by a diagram D equipped with a special vertex x_0, we can insert u in K near x_0 and we get a well-defined element $f(v)$ in the module $F_k(4)$.



But we have in $\widetilde{F}_k(4)$:

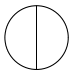
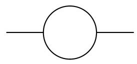


and that implies in $F_k(4)$:



Therefore $2u$ lies in Λ_k. $\square$

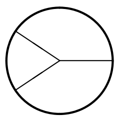
Corollary 4.6. *Suppose* 6 *is invertible in k. Then the modules* $F'_k(0)$ *and* $F'_k(2)$ *are free* Λ_k*-modules of rank one generated by:*


and


respectively.

Proof. Since $\mathfrak{S}_3$ is a group of order 6, the identity induces an isomorphism from $F_k(3)^-$ to $F_k(3) \otimes_{\mathfrak{S}_3} k^-$ and the corollary follows easily. $\square$

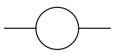
Corollary 4.7. *Let u be the primitive element of* $\mathcal{A} = \mathcal{A}(S^1, \emptyset)$ *represented by the diagram:*



Then the map $\lambda \mapsto \lambda u$ *from* Λ *to the module* $\mathcal{P}$ *of primitives of* $\mathcal{A}$ *is injective.*

Proof. Because of Proposition 1.2, $\mathcal{P} = \mathcal{P}_{\mathbf{Q}}$ contains the module

$$\mathrm{H}_0(\mathfrak{S}_2, F'(2)) \simeq F'(2) \otimes_{\mathfrak{S}_2} \mathbf{Q} \simeq F'(2).$$

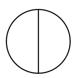
But u corresponds via these isomorphisms to the diagram  and the result follows. $\square$

It is not clear that Λ_k is commutative, but it is almost the case. If α and β are elements in Λ_k, and u an element of a module $\mathcal{A}^s_k(\Gamma, X)$ represented by a diagram with at least two 3-valent vertices outside of Γ, we may construct $\alpha\beta u$ by using α and β modifications near two different vertices. Therefore: $\alpha\beta u = \beta\alpha u$.

Proposition 4.8. *The algebra* Λ_k *has the following properties:*

$$\forall \alpha, \beta, \gamma \in \Lambda_k, \qquad \partial^\circ\gamma > 0 \Rightarrow \alpha\beta\gamma = \beta\alpha\gamma,$$
$$\forall \alpha, \beta \in \Lambda_k, \qquad 12\alpha\beta = 12\beta\alpha.$$

Proof. The first formula is a special case of the property explained above. For the second one, just use that property where u is represented by the diagram

$\Theta =$ 

in $F'_k(0)$ and remark that the composite: $\Lambda_k \to F_k(3) \otimes_{\mathfrak{S}_3} k^- \to F'_k(0)$ has a kernel annihilated by $6 \times 2 = 12$. $\square$

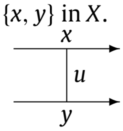
Corollary 4.9. *The algebra* Λ *is commutative.*

Proposition 4.10. *Let $\widehat{\Lambda}$ be the algebra Λ completed by the degree (i.e. $\widehat{\Lambda} = \prod_i \Lambda_i$). Let M be a 3-dimensional homology sphere. Then there is a unique element $\theta(M)$ in $\widehat{\Lambda}$ such that the LMO invariant of M is the exponential of the element $\theta(M)\Theta$.*

Proof. Let u be the LMO invariant of M constructed by Le–Murakami–Ohtsuki [LMO]. Then u is a group-like element in the completion of the module generated by 3-valent diagrams. Therefore its logarithm is primitive and lies in the completion of the module $F'(0)$. Since this module is a free $\widehat{\Lambda}$-module generated by Θ the result follows. □

5. Constructing elements in Λ

Let Γ be a curve and Z be a finite set. Let D be a (Γ, Z)-diagram. Let X be a finite set in D outside the set of vertices of D. Suppose that D is oriented near X. For each $x \neq y$ in X we have a diagram D_{xy} obtained from D by adding an edge u joining x and y in D. Cyclic orderings near x and y are chosen by an immersion from D_{xy} to the plane which is injective on a neighborhood of u and sends neighborhoods of x and y in K to horizontal lines with the same orientation and u to a vertical segment. This diagram D_{xy} depends only on the subset $\{x, y\}$ in X.



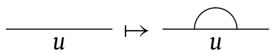
The sum of the diagrams D_{xy} for all subsets $\{x, y\} \subset X$ will be denoted by D_X.

Lemma 5.1. *Let Γ and Γ' be closed curves. Let X, Y and Z be finite disjoint sets. Let D be a $(\Gamma, X \cup Y)$-diagram and D' be a $(\Gamma', X \cup Y \cup Z)$ diagram. Suppose that the union H of D and D' over $X \cup Y$ lies in $\mathcal{D}^s(\Gamma \cup \Gamma', Z)$. The diagram H is oriented near X and Y by going from D' to D near X and from D to D' near Y. Then we have the following formula in $\mathcal{A}^s_k(\Gamma \cup \Gamma', Z)$:*

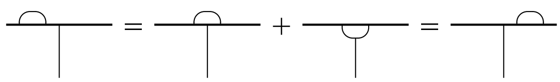
$$H_X - ptH = H_Y - qtH$$

where $p = \#X$, $q = \#Y$.

Proof. Set $\Gamma_1 = \Gamma \cup \Gamma'$. Let u be a point in H which is not a vertex. By adding one edge to H near u we get a new diagram H_u:



The class $[H_u]$ of H_u in $\mathcal{A}^s_k(\Gamma_1, Z)$ will be denoted by $\varphi(u)$. If u is not in Γ_1, $\varphi(u)$ is equal to $2t[H]$. Otherwise $\varphi(u)$ depends only on the component of Γ_1 which contains u:



Consider a map f from H to the circle $S^1 = \mathbf{R} \cup \{\infty\}$ satisfying the following:
– f is smooth and generic on Γ_1 and on each edge of K
– every singular value of f is the image of a unique critical point in an open edge of $H \setminus \Gamma_1$ or a unique vertex of H
– a vertex in H is never a local extremum of f
– each critical point of $f|\Gamma_1$ is not a vertex of H
– $f^{-1}(0) = X$, $\quad f^{-1}(1) = Y$, $\quad f^{-1}([0, 1]) = D$.

Let v be a regular value of f and V be the set $f^{-1}(v)$. The map f induces an orientation of H near each point of V. So $[H]_V$ is well defined in $\mathcal{A}^s_k(\Gamma_1, Z)$ and we can set:

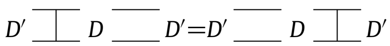
$$g(v) = [H_V] - 1/2 \sum_{u \in V} \varphi(u).$$

This expression is well defined because V meets every component of Γ_1 in a even number of points.

By construction we have: $g(v) = [H_X] - pt[H]$ if v is near 0 and $g(v) = [H_Y] - qt[H]$ if v is near 1. Then the last thing to do is to prove that g has no jump on the critical values of f.

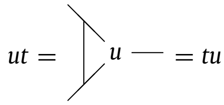
If v is the image of a critical point in an open edge in H, the jump of f in v is 0 because of the AS relations. If v is the image of a vertex in H, the jump is also 0 because of the IHX relations. Therefore the map g is constant and the lemma is proven. □

A special case of this lemma is the following equality:

 in $\mathcal{A}_k^s(\Gamma, Z)$

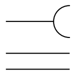
Corollary 5.2. *The element t is central in Λ_k.*

Proof. For every $u \in \Lambda_k$, we have:



□

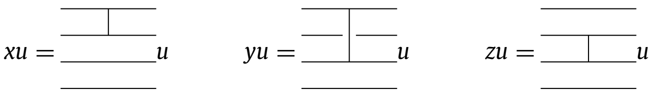
Let Γ_4 be the normal subgroup of order 4 of $\mathfrak{S}_4$. Consider the element $\delta \in {}_3\Delta_{k4}$ represented by the following diagram:



By gluing from the left or the right, we get a map $u \mapsto u\delta$ from $F_k(3)$ to $F_k(4)$ or a map $u \mapsto \delta u$ from $F_k(4)$ to $F_k(3)$. Denote by E the submodule of $F_k(4)$ of all elements $u \in F_k(4)$ satisfying the following conditions:

$$\forall \sigma \in \mathfrak{S}_4, \quad \delta\sigma u \in \Lambda_k \quad \text{and} \quad \forall \sigma \in \Gamma_4, \qquad \sigma u = u.$$

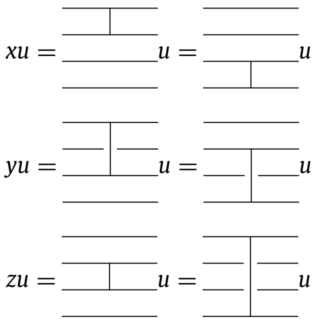
For every $u \in F_k(4)$, define elements xu, yu, zu by:



Proposition 5.3. *The module E is a graded $\Lambda_k[\mathfrak{S}_4]$-submodule of $F_k(4)$ and for every $u \in E$ we have:*

$$xu, yu, zu \in E, \qquad xu + yu + zu = 2tu.$$

Proof. The fact that E is a graded $\Lambda_k[\mathfrak{S}_4]$-submodule of $F_k(4)$ is obvious. Let u be an element of $F_k(4)$. Because of Lemma 5.1, we have:

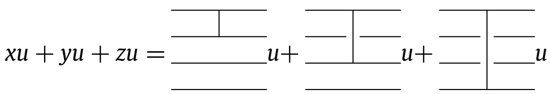


Hence, if σ is a permutation in $\mathfrak{S}_4$, there exists an element $\theta \in \{x, y, z\}$ such that $\sigma xu = \theta\sigma u$. More precisely $\mathfrak{S}_4$ acts on the set $\{x, y, z\}$ via an epimorphism $\sigma \mapsto \widehat{\sigma}$ from $\mathfrak{S}_4$ to $\mathfrak{S}_3$, and we have:

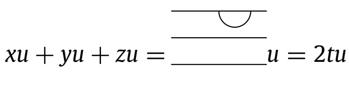
$$\sigma xu = \widehat{\sigma}(x)\sigma u, \qquad \sigma yu = \widehat{\sigma}(y)\sigma u, \qquad \sigma zu = \widehat{\sigma}(z)\sigma u.$$

The kernel of this epimorphism is Γ_4.

We have:

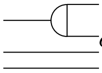


Because of Lemma 3.3, we have:



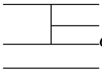
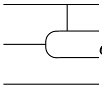
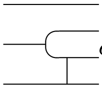
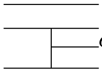
Moreover, if $u \in F_k(4)$ is Γ_4-invariant, xu, yu, zu are Γ_4-invariant too, and the last thing to do is to prove that $\delta x\sigma u, \delta y\sigma u, \delta z\sigma u$ are in Λ_k for every $u \in E$.

We have:

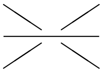
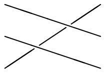
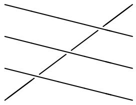
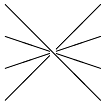
$\delta x\sigma u =$  $\sigma u = t\delta\sigma u \in \Lambda_k$

$$\delta y\sigma u = 2t\delta\sigma u - \delta x\sigma u - \delta z\sigma u$$

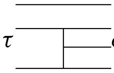
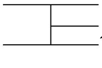
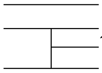
and it is enough to prove that $\delta z\sigma u$ belongs to Λ_k. Because of Lemma 5.1 we have:

$\delta z\sigma u =$  $\sigma u =$  $\sigma u =$  $\sigma u =$  σu

Let s, τ, τ', θ be the permutations in $\mathfrak{S}_4$ or $\mathfrak{S}_3$ represented by the following diagrams:

$s =$  $\tau =$  $\tau' =$  $\theta =$ 

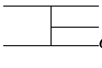
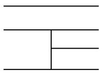
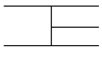
We have:

$\tau\delta z\sigma u = \tau$  $\sigma u =$  $\tau'\sigma u =$  $\tau'\sigma u$

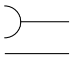
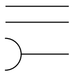
and then:

$$\tau\delta z\sigma u = \delta z\tau'\sigma u \qquad \Rightarrow \qquad \tau^2\delta z\sigma u = \delta z\tau'^2\sigma u.$$

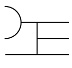
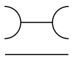
But τ'^2 lies in Γ_4 and $\tau^2\delta z\sigma u = \delta z\sigma u$. Therefore $\delta z\sigma u$ is invariant under cyclic permutations. We have also:

$s\delta z\sigma u = s$  $\sigma u = -$  $\theta\sigma u = -$  $\theta\sigma u$

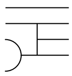
Since θ lies in Γ_4 also, $s\delta z\sigma u = -\delta z\sigma u$ and $\delta z\sigma u$ belongs to the submodule $F_k(3)^-$ of $F_k(3)$. Consider the following diagrams:

$\delta' =$  $\delta'' =$ 

We have to prove the last equality: $\delta'\delta z\sigma u = \delta''\delta z\sigma u$. Denote by σ_{ij} the transposition $i \leftrightarrow j$. We have:

$\delta'\delta z\sigma u =$  $\sigma u =$  $z\sigma u = (1 - \sigma_{12})xz\sigma u$

and similarly:

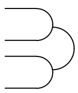
$\delta''\delta z\sigma u =$  $\sigma u = (1 - \sigma_{34})xz\sigma u$

But σ_{12} and σ_{34} are the same modulo Γ_4 and induce the transposition $y \leftrightarrow z$. Then we have:

$$\delta''\delta z\sigma u = xz\sigma u - xy\sigma_{34}\sigma u = xz\sigma u - xy\sigma_{12}\sigma u = \delta'\delta z\sigma u$$

and that finishes the proof. $\square$

Consider the following element of $F_k(4)$:

$a =$ 

For every $p > 0$ set: $x_p = \delta z^{p-1}a$. Because of the last result, x_p is an element of degree p in Λ_k. It is not difficult to check the following:

$$x_1 = 2t \qquad x_2 = t^2 \qquad 3x_4 = 4tx_3 + t^4$$

and Λ_k is freely generated in degree < 6 by:

$$1,\ t,\ t^2,\ t^3,\ t^4,\ t^5,\ x_3,\ \frac{tx_3 - t^4}{3},\ \frac{t^2x_3 - t^5}{3},\ \frac{x_5 + t^2x_3}{2}.$$

Let τ be a permutation in $\mathfrak{S}_4$ inducing the cyclic permutation $x \mapsto y \mapsto z \mapsto x$. Set: $z_1 = x, z_2 = y, z_3 = z, \alpha_1 = a$, $\alpha_2 = \tau a, \alpha_3 = \tau^2 a$. The group $\mathfrak{S}_3$ acts on E and for every $\sigma \in \mathfrak{S}_3$, every $i \in \{1, 2, 3\}$ and every $u \in E$ we have:

$$\sigma(z_i u) = z_{\sigma(i)}\sigma(u)$$
$$\sigma(\alpha_i) = \varepsilon(\sigma)\alpha_{\sigma(i)}$$

where $\varepsilon(\sigma)$ is the signature of σ. Denote also by f_1 the morphism $u \mapsto \delta u$ from E to Λ_k. If σ is the transposition keeping 1 fixed, on has for every $u \in E$:

$$f_1(\sigma(u)) = -f_1(u).$$

Therefore there are unique morphisms f_2 and f_3 from E to Λ_k such that:

$$f_{\sigma(i)}(\sigma(u)) = \varepsilon(\sigma)f_i(u)$$

for every $u \in E, \sigma \in \mathfrak{S}_3$ and $i \in \{1, 2, 3\}$. Moreover, if σ_i is the transposition keeping i fixed we have:

$$z_i(u - \sigma_i(u)) = f_i(u)\alpha_i$$

for every $u \in E$.

The set $\{1, 2, 3\}$ is canonically oriented and for every i, j and k distinct in $\{1, 2, 3\}$, there is a sign $i{\wedge}j{\wedge}k$ in $\{\pm 1\}$: the signature of the permutation $1 \mapsto i, 2 \mapsto j, 3 \mapsto k$.

Proposition 5.4. *Suppose* 6 *is invertible in k. Then there exist unique elements e, ε_p and $\beta_{i,p}$ in E, for $i \in \{1, 2, 3\}$ and $p \geq 0$ and unique elements ω_p ($p \geq 0$) in Λ_k such that the following holds for every $\sigma \in \mathfrak{S}_3$, every i, j, k distinct in $\{1, 2, 3\}$ and every $p \geq 0$:*

$$\beta_{1,p} + \beta_{2,p} + \beta_{3,p} = 0$$
$$\sigma(e) = e, \qquad \sigma(\varepsilon_p) = \varepsilon_p, \qquad \sigma(\beta_{i,p}) = \varepsilon(\sigma)\beta_{\sigma(i),p}$$
$$f_i(\alpha_i) = 2t, \qquad f_i(\beta_{i,p}) = 2\omega_p$$
$$f_i(\alpha_j) = -t, \qquad f_i(\beta_{j,p}) = -\omega_p$$
$$f_i(e) = f_i(\varepsilon_p) = 0$$
$$z_i\alpha_i = t\alpha_i, \qquad z_i\beta_{i,p} = \omega_p\alpha_i$$
$$z_i\alpha_j = i{\wedge}j{\wedge}k\, e + \frac{t}{3}(\alpha_j - \alpha_i)$$
$$z_i e = \frac{2t}{3}e + i{\wedge}j{\wedge}k\Big(\frac{10t^2}{9}(\alpha_j - \alpha_k) - \frac{1}{2}(\beta_{j,0} - \beta_{k,0})\Big)$$
$$z_i\beta_{j,p} = i{\wedge}j{\wedge}k\, \varepsilon_p + \frac{2t}{3}(\beta_{j,p} - \beta_{k,p}) + \omega_p\alpha_k$$
$$z_i\varepsilon_p = \frac{2t}{3}\varepsilon_p + i{\wedge}j{\wedge}k\Big(\frac{4t^2}{9}(\beta_{j,p} - \beta_{k,p}) - \frac{1}{2}(\beta_{j,p+1} - \beta_{k,p+1}) + \frac{2t\omega_p}{3}(\alpha_j - \alpha_k)\Big).$$

Proof. Consider formal elements ω'_p, for $p \geq 0$ of degree $3 + 2p$. Then $R = k[t, \omega'_0, \omega'_1, \ldots]$ is a graded algebra. Let E' be the R-module generated by elements α'_i, $\beta'_{i,p}$, e' and ε'_p (for $p \geq 0$ and $i \in \{1, 2, 3\}$) with the following relations:

$$\sum_i \alpha'_i = 0, \qquad \forall p \geq 0,\ \sum_i \beta'_{i,p} = 0.$$

This module is graded by:

$$\partial^\circ\alpha'_i = 0, \qquad \partial^\circ\beta'_{i,p} = 2 + 2p, \qquad \partial^\circ e' = 1, \qquad \partial^\circ\varepsilon'_p = 3 + 2p.$$

The symmetric group $\mathfrak{S}_3$ acts on E' by:

$$\sigma(\alpha'_i) = \varepsilon(\sigma)\alpha'_{\sigma(i)}, \qquad \sigma(\beta'_{i,p}) = \varepsilon(\sigma)\beta'_{\sigma(i),p}, \qquad \sigma(e') = e', \qquad \sigma(\varepsilon'_p) = \varepsilon'_p$$

and E' is a graded $R[\mathfrak{S}_3]$-module.

Using relations above we have well-defined maps $u \mapsto z_i u$ from E' to E' and the sum of these maps is $2t$. We have also linear maps f_i from E' to R sending e' and ε'_p to 0 and defined on the other generators by:

$$f_i(\alpha'_i) = 2t \qquad f_i(\beta'_{i,p}) = 2\omega'_p$$
$$f_i(\alpha'_j) = -t \qquad f_i(\beta'_{j,p}) = -\omega'_p.$$

It is not difficult to check the formula:

$$\forall u \in E', \qquad z_i(u - \sigma_i(u)) = f_i(u)\alpha'_i.$$

So the last thing to do is to construct an algebra homomorphism ψ from R to Λ_k and a morphism φ from E' to E which is linear over ψ sending α'_i to α_i and z_i to z_i.

Consider the elements $u(i, j, k) = z_i\alpha_j - t/3\alpha_j + t/3\alpha_i$ in E (for i, j, k distinct). One has:

$$\begin{aligned} u(i, j, k) - u(j, k, i) &= z_i\alpha_j - z_j\alpha_k - t/3(\alpha_j - \alpha_i - \alpha_k + \alpha_j) = z_i\alpha_j - z_j\alpha_k - t\alpha_j \\ &= z_i\alpha_j + (z_i + z_k - 2t)\alpha_k - t\alpha_j = z_i(\alpha_j + \alpha_k) + z_k\alpha_k - 2t\alpha_k - t\alpha_j \\ &= -z_i\alpha_i + z_k\alpha_k + t\alpha_i - t\alpha_k = 0. \end{aligned}$$

Then $u(i, j, k)$ is invariant under cyclic permutations. One has also:

$$u(i, j, k) + u(k, j, i) = (z_i + z_k)\alpha_j - 2t/3\alpha_j + t/3(\alpha_i + \alpha_k) = (2t - z_j)\alpha_j - t\alpha_j = 0.$$

Therefore $u(i, j, k)$ is totally antisymmetric in i, j, k and $i{\wedge}j{\wedge}k\, u(i, j, k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

$$e = \varphi(e') = i{\wedge}j{\wedge}k\, u(i, j, k).$$

The element $v(i, j, k) = i{\wedge}j{\wedge}k(z_je - z_ke)$ is clearly symmetric under the transposition $j \leftrightarrow k$. So it depends only on i and we can set:

$$\beta_{i,0} = \frac{20t^2}{9}\alpha_i + \frac{2}{3}v(i, j, k).$$

Hence we have:

$$\begin{aligned} z_ie &= \frac{1}{3}(2z_i - z_j - z_k + 2t)e = \frac{2t}{3}e + \frac{i{\wedge}j{\wedge}k}{3}(v(k, i, j) - v(j, k, i)) \\ &= \frac{2t}{3}e + i{\wedge}j{\wedge}k\Big(\frac{10t^2}{9}(\alpha_j - \alpha_k) - \frac{1}{2}(\beta_{j,0} - \beta_{k,0})\Big). \end{aligned}$$

It is easy to see that the sum of the $\beta_{i,0}$ vanishes and we can set: $\varphi(\beta'_{i,0}) = \beta_{i,0}$. On the other hand we have:

$$f_i(-\beta_{k,0}) = -f_i(\beta_{j,0})$$

and $f_i(\beta_{j,0})$ depends only on i. But we have: $f_i(\beta_{j,0}) = f_j(\beta_{k,0})$ and $f_i(\beta_{j,0})$ does not depend on i. So we can set: $\omega_0 = -f_i(\beta_{j,0})$. Since $\beta_{i,0} + \beta_{j,0} + \beta_{k,0}$ is trivial, we have also: $f_i(\beta_{i,0}) = 2\omega_0$ and we can set: $\psi(\omega'_0) = \omega_0$.

Set: $w(i, j, k) = z_i\beta_{j,0} - \frac{2t}{3}(\beta_{j,0} - \beta_{k,0}) - \omega_0\alpha_k$. One has:

$$\begin{aligned} w(i, j, k) - w(j, k, i) &= z_i\beta_{j,0} - z_j\beta_{k,0} - \frac{2t}{3}(-3\beta_{k,0}) - \omega_0(\alpha_k - \alpha_i) \\ &= z_i\beta_{j,0} - (2t - z_i - z_k)\beta_{k,0} + 2t\beta_{k,0} - \omega_0(\alpha_k - \alpha_i) \\ &= z_i(\beta_{j,0} + \beta_{k,0}) + z_k\beta_{k,0} - \omega_0(\alpha_k - \alpha_i) \\ &= f_i(\beta_{j,0})\alpha_i + 1/2f_k(\beta_{k,0})\alpha_k - \omega_0(\alpha_k - \alpha_i) = 0. \end{aligned}$$

Then $w(i, j, k)$ is invariant under cyclic permutations. One has also:

$$\begin{aligned} w(i, j, k) + w(k, j, i) &= z_i\beta_{j,0} + z_k\beta_{j,0} - \frac{2t}{3}(3\beta_{j,0}) - \omega_0(\alpha_k + \alpha_i) \\ &= (2t - z_j)\beta_{j,0} - 2t\beta_{j,0} + \omega_0\alpha_j = -z_j\beta_{j,0} + \omega_0\alpha_j = 0. \end{aligned}$$

Therefore $w(i, j, k)$ is totally antisymmetric in i, j, k and $i{\wedge}j{\wedge}k\, w(i, j, k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

$$\varepsilon_0 = \varphi(\varepsilon'_0) = i{\wedge}j{\wedge}k\, w(i, j, k).$$

Let $p \geq 0$ be an integer. Suppose that $\beta_{i,q}$ and ε_q are constructed for $q \leq p$ and φ and ψ are constructed in degrees $\leq 3+2p$. Consider the element $u(i, j, k) = i{\wedge}j{\wedge}k(z_j - z_k)\varepsilon_p + \frac{4t^2}{3}\beta_{i,p} + 2t\omega_p\alpha_i$. This element is invariant under the transposition $j \leftrightarrow k$ and depends only on i. So we can set:

$$\beta_{i,p+1} = \frac{2}{3}u(i, j, k).$$

It is easy to check the following:

$$\beta_{1,p+1} + \beta_{2,p+1} + \beta_{3,p+1} = 0$$
$$z_i\varepsilon_p = \frac{2t}{3}\varepsilon_p + i\wedge j\wedge k\Big(\frac{4t^2}{9}(\beta_{j,p} - \beta_{k,p}) - \frac{1}{2}(\beta_{j,p+1} - \beta_{k,p+1}) + \frac{2t\omega_p}{3}(\alpha_j - \alpha_k)\Big)$$

and we can set: $\varphi(\beta'_{i,p+1}) = \beta_{i,p+1}$. On the other hand we have:

$$f_i(-\beta_{k,p+1}) = -f_i(\beta_{j,p+1})$$

and $f_i(\beta_{j,p+1})$ depends only on i. But we have: $f_i(\beta_{j,p+1}) = f_j(\beta_{k,p+1})$ and $f_i(\beta_{j,p+1})$ does not depend on i. So we can set: $\omega_{p+1} = -f_i(\beta_{j,p+1})$. Since $\beta_{i,p+1}+\beta_{j,p+1}+\beta_{k,p+1}$ is trivial, we have also: $f_i(\beta_{i,p+1}) = 2\omega_{p+1}$ and we can set: $\psi(\omega'_{p+1}) = \omega_{p+1}$.

Set: $w(i,j,k) = z_i\beta_{j,p+1} - \frac{2t}{3}(\beta_{j,p+1} - \beta_{k,p+1}) - \omega_{p+1}\alpha_k$. One has:

$$\begin{aligned} w(i,j,k) - w(j,k,i) &= z_i\beta_{j,p+1} - z_j\beta_{k,p+1} - \frac{2t}{3}(-3\beta_{k,p+1}) - \omega_{p+1}(\alpha_k - \alpha_i) \\ &= z_i\beta_{j,p+1} - (2t - z_i - z_k)\beta_{k,p+1} + 2t\beta_{k,p+1} - \omega_{p+1}(\alpha_k - \alpha_i) \\ &= z_i(\beta_{j,p+1} + \beta_{k,p+1}) + z_k\beta_{k,p+1} - \omega_{p+1}(\alpha_k - \alpha_i) \\ &= f_i(\beta_{j,p+1})\alpha_i + 1/2 f_k(\beta_{k,p+1})\alpha_k - \omega_{p+1}(\alpha_k - \alpha_i) = 0. \end{aligned}$$

Then $w(i,j,k)$ is invariant under cyclic permutations. One has also:

$$\begin{aligned} w(i,j,k) + w(k,j,i) &= z_i\beta_{j,p+1} + z_k\beta_{j,p+1} - \frac{2t}{3}(3\beta_{j,p+1}) - \omega_{p+1}(\alpha_k + \alpha_i) \\ &= (2t - z_j)\beta_{j,p+1} - 2t\beta_{j,p+1} + \omega_{p+1}\alpha_j = -z_j\beta_{j,p+1} + \omega_{p+1}\alpha_j = 0. \end{aligned}$$

Therefore $w(i,j,k)$ is totally antisymmetric in i,j,k and $i\wedge j\wedge k\; w(i,j,k)$ is invariant under the action of $\mathfrak{S}_3$. So one can set:

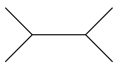
$$\varepsilon_{p+1} = \varphi(\varepsilon'_{p+1}) = i\wedge j\wedge k\; w(i,j,k).$$

So φ and ψ are defined by induction and the result follows. $\square$

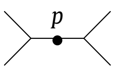
Remark. The subalgebra Λ' of Λ_k generated by the x_i's is generated by $x_1, x_3, x_5, \ldots$ and also by $t, \omega_0, \omega_1, \ldots$. Then every x_i can be expressed in term of t and the ω_j's. In low degree we get:

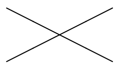
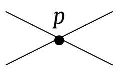
$$x_1 = 2t, \qquad x_2 = t^2, \qquad x_3 = 4t^3 - \frac{3}{2}\omega_0, \qquad x_4 = 5t^4 - 2t\omega_0,$$
$$x_5 = 12t^5 - \frac{17}{2}t^2\omega_0 + \frac{3}{2}\omega_1, \qquad x_6 = 21t^6 - 17t^3\omega_0 + 5t\omega_1 - \frac{3}{2}\omega_0^2,$$
$$x_7 = 44t^7 - \frac{91}{2}t^4\omega_0 - \frac{7}{2}t\omega_0^2 + \frac{37}{2}t^2\omega_1 - \frac{3}{2}\omega_2.$$

Suppose that $\alpha_i \in E$ is represented by:

$\alpha_i =$ 

Then we set:



$= \beta_{i,p}$,  $= e$, 

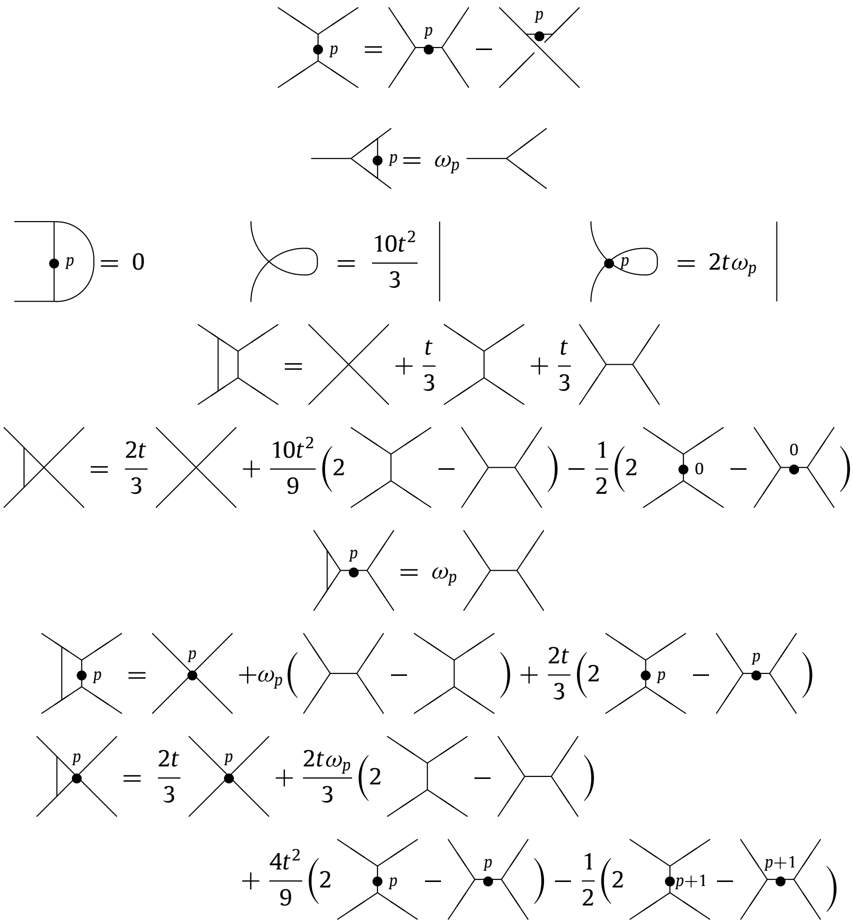
$= \varepsilon_p$

These diagrams are well defined in $F_k(4)$ if 6 is invertible in k. By gluing we are able to define new (Γ, X)-diagrams represented by a graph D containing Γ such that:

— the set ∂D of 1-valent vertices of D is the disjoint union of $\partial\Gamma$ and X
— each vertex of D in $\Gamma \setminus \partial\Gamma$ is 3-valent
— each vertex of D is 1-valent, 3-valent, or 4-valent
— each 3-valent vertex of D is oriented (by a cyclic ordering)
— some 4-valent vertex is marked by a bullet and labeled by a nonnegative integer
— some edge is marked by a bullet and labeled by a nonnegative integer
— each marked edge is outside of Γ and its boundary has two 3-valent vertices
— the marked edges are pairwise disjoint.

Such a diagram will be called an extended (Γ, X)-diagram. Each extended (Γ, X)-diagram is a linear combination of usual (Γ, X)-diagrams. A marked diagram D is an extended diagram with at least one marqued vertex or one marked edge. The sum of the markings is called the total marking of D.

Proposition 5.5. *Suppose* 6 *is invertible in k. Then we have the following formulas:*



for every $p \geq 0$.

Proof. This is essentially a graphical version of Proposition 5.4. □

There are many relations in the algebra Λ. Kneissler [14] founded relations in term of the x_i's. In term of the ω_i's Kneissler's result becomes the following:

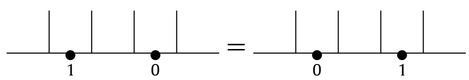
Theorem 5.6. *The following relations hold in* Λ*:*

$$\forall p, q \geq 0, \qquad \omega_p \omega_q = \omega_0 \omega_{p+q}.$$

Theorem 5.7. *Let* Γ *be a closed curve and* X *be a finite set. Let* u *be an element of* $\mathcal{A}^s(\Gamma, X)$ *represented by a marked diagram* D *with total marking* p*. Let* D_0 *be the diagram* D *where each marking is replaced by* 0*. Then* u *depends only on* p *and* D_0*. Moreover* $\omega_q u$ *depends only on* $p + q$ *and* D_0*.*

Proof. Here we are working over the rationals ($k = \mathbf{Q}$).

Lemma 5.7.1. *The following relation holds in* $F(6)$*:*

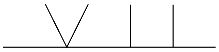


Proof. Let E_n be the component of $F(6)$ of degree n. These modules can be determined by computer for $n \leq 6$. In this range the dimensions are:

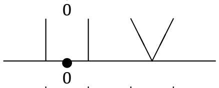
24 60 120 199 309 439 594

The desired relation lies in the module E_6 and can be checked directly. More precisely, E_n decomposes into a direct sum of pieces corresponding to the Young diagrams of size 6. Using this decomposition and formulas in Proposition 5.5 we get:

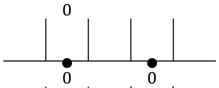
 $= A_0(4, 2) + A_0(2, 2, 2) + A_0(3, 1, 1, 1)$

 $= A_1(4, 2) + A_1(3, 2, 1)$

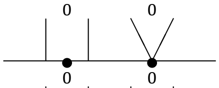
$= A_2(4, 2) + A_2(2, 2, 2) + A_2(3, 1, 1, 1) + A_2(3, 2, 1)$



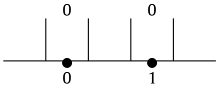
$= A_3(4, 2) + A_3(3, 2, 1) + A_3(5, 1)$



$= A_4(4, 2) + A_4(2, 2, 2) + A_4(3, 1, 1, 1) + A_4(3, 2, 1)$

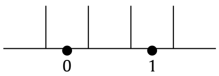


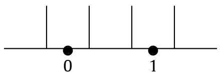
$= A_5(4, 2) + A_5(5, 1) + A_5(3, 2, 1)$

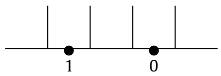


$= A_6(4, 2) + A_6(2, 2, 2) + A_6(3, 1, 1, 1) + A_6(3, 2, 1)$

It is not difficult to see that the symmetry σ along a vertical axis acts trivially on $A_6(4, 2), A_6(2, 2, 2), A_6(3, 1, 1, 1), A_6(3, 2, 1)$ and then on the last diagram. So we have:

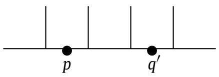


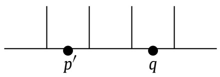
$= \sigma$


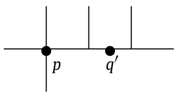
$=$


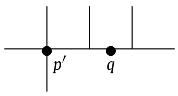
and that proves the lemma. □

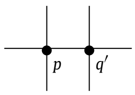
Lemma 5.7.2. *For every p and q we have the following relations in $F(6)$:*

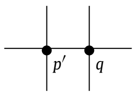


$=$




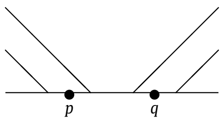
$=$


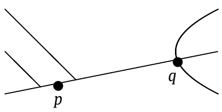


$=$


with $p' = p + 1$ and $q' = q + 1$.

Proof. Consider the following diagrams:

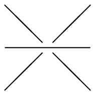
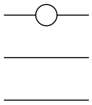
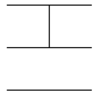
$A(p, q) =$ 

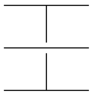
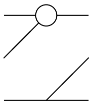
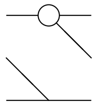
$B(p, q) =$ 

$C(p, q) =$ 

These diagrams are morphisms in the category Δ.

Consider the following morphisms in this category:

$\tau =$  $\theta =$  $\alpha =$ 

$\beta =$  $u =$  $v =$ 

Then we set:

$$f = \left(\frac{1}{2}\alpha - \frac{1}{6}\theta\right) \circ (1+\tau)$$
$$g = (1+\tau) \circ \left(\frac{1}{2}\alpha - \frac{1}{6}\theta\right)$$
$$\varphi = \frac{2}{3}f \circ (\beta - \alpha) + \frac{2}{9}\theta^2 + \frac{2}{3}u$$
$$\psi = \frac{2}{3}(\beta - \alpha) \circ g + \frac{2}{9}\theta^2 + \frac{2}{3}v.$$

Because of Proposition 5.5 we can check the following:

$$A(p,q) \circ f = B(p,q), \qquad A(p,q) \circ \varphi = A(p,q+1),$$
$$g \circ B(p,q) = C(p,q), \qquad \psi \circ A(p,q) = A(p+1,q).$$

Because of Lemma 5.7.1 we have: $A(0,1) = A(1,0)$. Therefore we get:

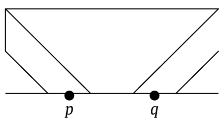
$$A(p,q+1) = \psi^p \circ A(0,1) \circ \varphi^q = \psi^p \circ A(1,0) \circ \varphi^q = A(p+1,q),$$
$$B(p,q+1) = A(p,q+1) \circ f = A(p+1,q) \circ f = B(p+1,q),$$
$$C(p,q+1) = g \circ B(p,q+1) = g \circ B(p+1,q) = C(p+1,q)$$

and that proves the lemma. □

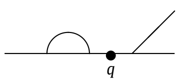
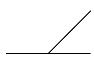
Then diagrams $A(p,q), B(p,q), C(p,q)$ depend only on $p+q$ and we can set:

$$A(p,q) = A(p+q), \qquad B(p,q) = B(p+q), \qquad C(p,q) = C(p+q).$$

Let u_0 be the diagram represented in $F(3)$ by $1 \in \Lambda \subset F(3)$. By gluing u_0 on the diagram $A(p,q)$ we get the following diagram v in $F(3)$:

$v =$ 

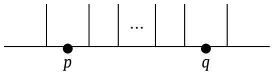
depending only on $p+q$. Because of Proposition 5.5, we get:

$v = \omega_p$  $= 2\omega_p\omega_q$ 

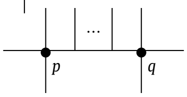
Therefore $\omega_p\omega_q$ depend only on $p+q$ and Theorem 5.6 follows.

Let D be a marked diagram representing an element U in $\mathcal{A}^s(\Gamma, X)$. Then $D \setminus \Gamma$ is connected. Let Z be the set of all marked vertices or edges of D. We will say that two elements u and v in Z are related if there is a path γ in D connecting u and v such that γ does not meet Γ and meets Z only in u and v. This relation generates an equivalence relation $\equiv$. Since $D \setminus \Gamma$ is connected, Z has only one equivalence class modulo $\equiv$. Therefore in order to prove the first part of Theorem 5.7 it is enough to prove the following: if u and v in Z are related the class of D in $\mathcal{A}^s(\Gamma, X)$ depends only on the sum of the marking of u and v.

Consider the following diagrams in $F(6+n)$, for some integers p, q, n:

$A_n(p, q) =$ 

$B_n(p, q) =$ 

$C_n(p, q) =$ 

If u and v in Z are related D contains a subdiagram isomorphic to $A_n(p, q)$, $B_n(p, q)$ or $C_n(p, q)$. Then it is enough to prove that $A_n(p, q)$, $B_n(p, q)$ and $C_n(p, q)$ depend only on n and $p + q$. Let X be one of the symbol A, B, C. Because of Lemma 3.3, we can push away all strands in the middle part of $X_n(p, q)$ through the marked edge (or the marked vertex) in the right part of the diagram and $X_n(p, q)$ is equivalent in $F(6 + n)$ to a linear combination of diagrams containing $X(p, q)$. Then, because of Lemma 5.7.2, $X_n(p, q)$ depends only on n and $p + q$ and the first part of Theorem 5.7 is proven.

The element $\omega_q U$ is represented by a diagram D' obtained from D by adding a new marked edge with marking q. Therefore $\omega_q U$ depends only on D_0 and the sum of q and the total marking of D. $\square$

Remark. Consider the commutative $\mathbf{Q}$-algebra R' defined by the following presentation:

– generators: $t, \omega_0, \omega_1, \ldots$

– relations: $\omega_p\omega_q = \omega_0\omega_{p+q}$, for every p, q.

We have a canonical morphism from R' to Λ. On the other hand there is a morphism $f : R' \longrightarrow \mathbf{Q}[t, \sigma, \omega]$ sending t to t and each ω_p to $\omega\sigma^p$. It is easy to see that this morphism is injective with image $R_0 = \mathbf{Q}[t] \oplus \omega\mathbf{Q}[t, \sigma, \omega]$. Then the morphism $R' \longrightarrow \Lambda_k$ induces a morphism from R_0 to Λ:

Proposition 5.8. *Let R be the polynomial algebra $\mathbf{Q}[t, \sigma, \omega]$ where t, σ and ω are formal variables of degree 1, 2 and 3 respectively and R_0 be the subalgebra $\mathbf{Q}[t] \oplus \omega\mathbf{Q}[t, \sigma, \omega]$ of R. Then there is a unique graded algebra homomorphism φ from R_0 to Λ sending t to t and each $\omega\sigma^p$ to ω_p.*

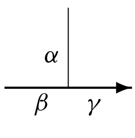
6. Detecting elements in Λ

In this section we will construct weight functions on modules of diagrams and characters on Λ using Lie superalgebras.

Let L be a finite-dimensional Lie superalgebra over a field K equipped with a nonsingular supersymmetric bilinear form $\langle\, ,\, \rangle$ invariant under the adjoint representation. Such a data will be called a quadratic Lie superalgebra and the bilinear form is called the inner form. Take a homogeneous basis (e_j) of L and its dual basis (e'_j). The Casimir element $\Omega = \sum_j e_j \otimes e'_j \in L \otimes L$ is independent of the choice of the basis and its degree is zero.

Let Γ be an closed oriented curve and $X = [n]$ be a finite set. Suppose that a L-representation E_i is chosen for each component Γ_i of Γ. We will say that Γ is colored by L-representations. Then it is possible to construct a linear map from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$ in the following way:

Let D be a (Γ, X)-diagram. Up to some changes of cyclic ordering we may as well suppose that, at each vertex x in Γ the cyclic ordering is given by (α, β, γ) where α is the edge which is not contained in Γ and β is the edge in Γ ending at x (with the orientation of Γ).



For each component Γ_i we can take a basis (e_{ij}) of E_i and its dual basis (e'_{ij}) of the dual E'_i of E_i and we get a Casimir element $\omega_i = \Sigma_j e_{ij} \otimes e'_{ij} \in E_i \otimes E'_i$. This element is of degree zero and is independent of the choice of the basis.

For each oriented edge α in D denote by $V(\alpha)$ the module L if α is not contained in Γ and E_i (resp. E'_i) if α is contained in the component Γ_i of Γ with a compatible (resp. not compatible) orientation. If α is an oriented edge in D denote by $W(\alpha)$ the module $V(\alpha) \otimes V(-\alpha)$ where $-\alpha$ is the edge α equipped with the opposite orientation.

Let a be an edge in D. Take an orientation of a compatible with the orientation of Γ if a is contained in Γ. Denote also by $\omega(a)$ the Casimir element ω if a is not contained in Γ and the element ω_i if a is contained in Γ_i. This element belongs to the module $W(a)$ and is independent on the orientation of a. If a numbering of the set of edges is chosen the tensor product $W = \otimes_a W(a)$ is well defined and the element $\Omega = \otimes_a \omega(a)$ is a well-defined element in W.

Let x be a 3-valent vertex in D. There are three oriented edges α, β and γ ending at x (the ordering (α, β, γ) is chosen to be compatible with the cyclic ordering given at x and, if x is in Γ, α is supposed to be outside of Γ).

Then we get a module $H(x) = V(\alpha) \otimes V(\beta) \otimes V(\gamma)$. If a numbering of the set of 3-valent vertices of D is chosen, the module $\otimes_x H(x)$ is well defined. We can permute (in the super sense) the big tensor product W and we get an isomorphism φ from W to the module:

$$H = L^{\otimes n} \otimes \otimes_x H(x)$$

and $\varphi(\Omega)$ is an element of H.

Suppose that x is not contained in Γ. Then the rule $u \otimes v \otimes w \mapsto \langle [u, v], w\rangle$ induces a map f_x from $H(x)$ to K. If x is in Γ the rule $u \otimes e \otimes f \mapsto (-1)^{\partial^\circ f \partial^\circ (u\otimes e)} f(ue)$ is a map f_x from $H(x)$ to K. Hence the image of $\varphi(\Omega)$ under the tensor product of all f_x is an element $\Phi_L(D) \in L^{\otimes n}$. Since elements w and w_i and maps f_x are of degree zero, this element does not depend on these numberings.

Since the map $u \otimes v \otimes w \mapsto \langle [u, v], w\rangle$ from $L \otimes L \otimes L$ to K is totally antisymmetric (in the super sense), $\Phi_L(D)$ is multiplied by -1 if one cyclic ordering is changed in D. Moreover, the Jacobi identity and the property of the L-action on modules E_i imply that the correspondence $D \mapsto \Phi_L(D)$ is compatible with the IHX relation. Therefore this correspondence induces a well-defined linear map Φ_L from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$.

Definition. A Lie superalgebra L over a field K will be called quasisimple if it satisfies the two conditions:
— L is not abelian
— every endomorphism of L of degree 0 is the multiplication by a scalar.

Remark. Every simple Lie superalgebra is quasisimple but the converse is not true.

Lemma. *A quasisimple quadratic Lie superalgebra has a trivial center and a surjective Lie bracket.*

Proof. Let L be a quasisimple quadratic Lie superalgebra over a field K. Let f be a morphism from L to K. By duality we get a morphism g from K to L. The composite $g \circ f$ is an endomorphism of L and there is a scalar $\lambda \in K$ such that: $g \circ f = \lambda \mathrm{Id}$.

Suppose $f \neq 0$. Then f is surjective, g is injective, $g \circ f$ is not trivial and $\lambda \neq 0$. Therefore $g \circ f$ is bijective and f is bijective also. But that is impossible because L is not abelian.

Then every morphism from L to K is zero and (by duality) every morphism from K to L is zero too. The result follows. □

Theorem 6.1. *Let K be a field and a k-algebra and L be a quasisimple quadratic Lie superalgebra over K. Then there is a well-defined character $\chi_L : \Lambda_k \longrightarrow K$ such that:*

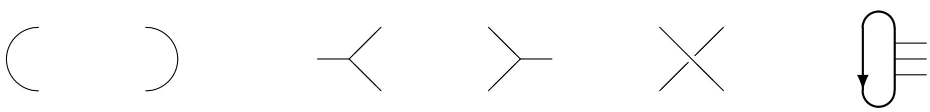
for every closed oriented curve Γ colored by L-representations and every finite set X, the map Φ_L satisfies the following property:

$$\forall \alpha \in \Lambda_k, \forall u \in \mathcal{A}^s(\Gamma, X), \quad \Phi_L(\alpha u) = \chi_L(\alpha)\Phi_L(u).$$

Let A be a k-subalgebra of K. Suppose K is the fraction field of A and A is a unique factorization domain. Suppose also that L contains a finitely generated A-submodule L_A such that the Lie bracket and its dual are defined on L_A. Then χ_L takes values in A.

Proof. First of all, it is possible to extend the map Φ_L to a functor between two categories $Diag(L)$ and $\mathcal{C}(L)$. The objects of these categories are the sets $[p]$, $p \geq 0$. For $p, q \geq 0$ the set of morphisms in $\mathcal{C}(L)$ from $[p]$ to $[q]$ is the set of L-linear homomorphisms from $L^{\otimes p}$ to $L^{\otimes q}$, and the set of morphisms in $Diag(L)$ from $[p]$ to $[q]$ is the k-module generated by the isomorphism classes of $(\Gamma, [p] \cup [q])$-diagrams where Γ is any L-colored oriented curve and where the relations are the AS and IHX relations.

These two categories are monoidal and $Diag(L)$ contains Δ_k as a subcategory. Moreover $Diag(L)$ is generated (as a monoidal category) by the following morphisms:



The last morphism is a morphism in $Diag(L)$ from $[p]$ to $[0]$ depending on an integer $p \geq 0$ and a L-representation E.

The map Φ_L associates to each L-colored $(\Gamma, [p] \cup [q])$-diagram D an element $\Phi_L(K)$ in $L^{\otimes p} \otimes L^{\otimes q}$. But $L^{\otimes p}$ is isomorphic to its dual and $\Phi_L(K)$ may be seen as a linear map from $L^{\otimes p}$ to $L^{\otimes q}$.

It is not difficult to see that the image under Φ_L of the generators above are:
— the inner form from $L^{\otimes 2}$ to $L^{\otimes 0} = K$,
— the Casimir element consider as a morphism from $K = L^{\otimes 0}$ to $L^{\otimes 2}$,

— the Lie bracket from $L^{\otimes 2}$ to L,
— the dual of the Lie bracket (the Lie cobracket) from L to $L^{\otimes 2}$,
— the map $x \otimes y \mapsto (-1)^{\partial^\circ x \partial^\circ y} y \otimes x$ from $L^{\otimes 2}$ to itself,
— the map $x_1 \otimes \cdots \otimes x_p \mapsto \tau_E(x_1 \ldots x_p)$ from $L^{\otimes p}$ to $L^{\otimes 0} = K$,
where $\tau_E(x_1 \ldots x_p)$ is the supertrace of the endomorphism $x_1 \ldots x_p$ of E.

All these maps are L-linear. Therefore Φ_L induces a functor still denoted by Φ_L from $Diag(L)$ to the category $\mathcal{C}(L)$.

Let Γ be a L-colored oriented curve and $X = [n]$ be a finite set. Consider an element $\alpha \in \Lambda_k$ and an element $u \in \mathcal{A}^s(\Gamma, X)$ represented by a (Γ, X)-diagram D. Take a 3-valent vertex x in D and a bijection from [3] to the set of edges ending at x. By taking off a neighborhood of x in D, we get a diagram H inducing a morphism v in $Diag(L)$ from [3] to $[n]$.

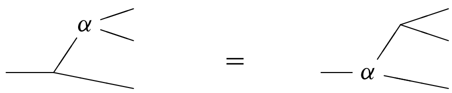
On the other hand, α induces a morphism β in $Diag(L)$ from [0] to [3], and $1 \in \Lambda$ induces an element β_0 from [0] to [3]. Let $\widetilde{u}$ and $\widetilde{\alpha u}$ be the morphisms from [0] to $[n]$ induced by u and αu. We have:

$$\widetilde{u} = v \circ \beta_0, \qquad \widetilde{\alpha u} = v \circ \beta.$$

Hence:

$$\Phi_L(\widetilde{u}) = \Phi_L(v) \circ \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \Phi_L(v) \circ \Phi_L(\beta).$$

The elements $\alpha \in \Lambda_k$ and $1 \in \Lambda_k$ also induce morphisms γ and γ_0 from [2] to [1]. Denote by φ and φ_0 the morphisms $\Phi_L(\gamma)$ and $\Phi_L(\gamma_0)$. The morphism φ_0 is the Lie bracket and φ is L-linear and antisymmetric. Since α belongs to Λ_k, we have the following:



and, for every x, y, z in L, we have: $[\varphi(x \otimes y), z] = \varphi([x, y] \otimes z)$.

Denote by $u \mapsto [u]$ the Lie bracket from $L^{\otimes 2}$ to L. For every $u \in L^{\otimes 2}$ and every $z \in L$ we have: $[\varphi(u), z] = \varphi([u] \otimes z)$.

Suppose $[u] = 0$ then $[\varphi(u), z]$ vanishes for every $z \in L$ and $\varphi(u)$ lies in the center of L. Since this center is trivial, $\varphi(u)$ is trivial too. Therefore $\varphi(u)$ depends only on the image $[u]$ of u. Since the Lie bracket is surjective, there is a unique morphism ψ from L to L such that:

$$\forall u \in L^{\otimes 2}, \quad \varphi(u) = \psi([u])$$

and there is a unique $\lambda \in K$ such that:

$$\forall u \in L^{\otimes 2}, \quad \varphi(u) = \lambda[u]$$

and we have:

$$\Phi_L(\beta) = \lambda \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \lambda \Phi_L(\widetilde{u}), \qquad \Phi_L(\alpha u) = \lambda \Phi_L(u).$$

Now it is easy to see that $\alpha \mapsto \lambda$ is a character depending only on L and the Casimir element Ω.

Suppose now that L contains a finitely generated A-submodule L_A such that the Lie bracket and the Lie cobracket (the dual of the Lie bracket) are defined on L_A. Let α be an element in Λ_k represented by a $(\emptyset, [3])$-diagram D and $u \in K$ be its image under χ_L. Because this diagram is connected there exists a continuous map f from D to $[0, 1]$ such that:
— $f(1) = f(2) = 0$ $f(3) = 1$
— f is affine and injective on each edge of D
— f is injective on the set of 3-valent vertices of D
— f has no local extremum.

Such a map can be constructed by induction on the number of edges of D. Using this map, the map from [2] to [1] represented by D can be described by composition, tensor product, Lie bracket and Lie cobracket and we have:

$$\forall x, y \in L_A, \quad u[x, y] \in L_A.$$

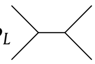
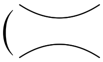
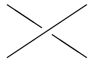
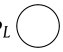
Let w be a nonzero element in the image of the Lie bracket $L_A \otimes L_A \longrightarrow L_A$. By applying the formula above for each power of α, we get:

$$\forall p \geq 0, \quad u^p w \in L_A.$$

Since L_A is finitely generated the A-submodule of K generated by the powers of u is also finitely generated. Then u lies in the integral closure of A in K. Since A is a unique factorization domain, A is integrally closed and u belongs to A. Therefore $\chi_L(\alpha)$ lies in A for every $\alpha \in \Lambda_k$. $\square$

Remark. If every endomorphism of L is the multiplication by a scalar, every invariant bilinear form of L is a multiple of the given inner form. If we divide the inner form par some $c \in K$, we multiply the Casimir element Ω by c and for every $\alpha \in \Lambda_k$ of degree n, $\chi_L(\alpha)$ is multiplied by c^n.

Proposition 6.2. *Let L be the Lie algebra sl_2 (defined over K). Then the functor Φ_L satisfies the following properties:*

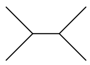
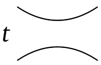
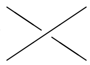
Φ_L  $= \Phi_L\, t\,\Big($  $-$  $\Big)$ $\qquad \Phi_L$  $= 3$

Moreover there is a unique graded algebra homomorphism χ_{sl2} from Λ_k to $k[t]$ sending t to t and each ω_n to 0 such that the character χ_L is the composite:

$$\Lambda_k \xrightarrow{\chi_{sl2}} k[t] \xrightarrow{\gamma} K$$

where γ is a k-algebra homomorphism. If the inner form on L send $\alpha \otimes \beta$ to the trace of $\alpha\beta$, γ sends t to 2.

Proof. Since L is defined over $\mathbf{Q}$ it is enough to consider the case $k = K = \mathbf{Q}$. Set:

$U =$  $- t$  $+ t$ 

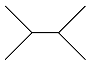
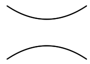
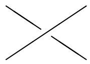
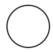
The element $\Phi_L(U)$ is a map from $L^{\otimes 2} = \Lambda^2(L) \oplus S^2(L)$ to itself. Since U is antisymmetric on the source and the target, $\Phi_L(U)$ is trivial on $S^2(L)$ and its image is contained in $\Lambda^2(L)$. Since L is 3-dimensional, the Lie bracket $\Lambda^2(L) \longrightarrow L$ is bijective. But U composed with this bracket is zero. Therefore U is killed by Φ_L.

The fact that Φ_L sends the circle to 3 come from the fact that L is 3-dimensional.

Denote by $\equiv$ the following relation:

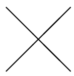
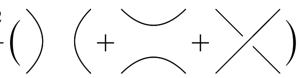
$$a \equiv b \Longleftrightarrow \Phi_L(a) = \Phi_L(b)$$

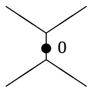
So we have:

 $\equiv t$  $- t$  $\qquad$ and $\qquad$  $\equiv 3$

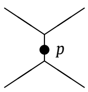
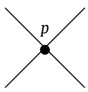
and it is easy to see by induction that every element α in Λ_k is equivalent to some polynomial $P(t) \in k[t]$. Let c be the scalar $\chi_L(t)$. Then we have: $\Phi_L(\alpha) = P(c)$. If α is homogeneous of degree n, we have: $P(t) = at^n$ and: $\Phi_L(\alpha) = ac^n$. Then $P(t)$ is completely determined by $\Phi_L(\alpha)$. Therefore $\alpha \mapsto P(t)$ is a well-defined algebra homomorphism χ_{sl2} from Λ_k to $k[t]$ such that χ_L is the composite $\gamma \circ \chi_{sl2}$ where γ sends t to c. If the inner form is $\alpha \otimes \beta \mapsto \tau(\alpha\beta)$, we have $c = 2$ and γ sends t to 2.

A direct computation gives the following:

 $\equiv \dfrac{2t^2}{3}$ 

 $\equiv 0\,.$

Then by induction we get the following for every $p \geq 0$:

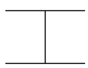
 $\equiv 0\,,$ $\qquad$  $\equiv 0\,,$ $\qquad \omega_p \equiv 0\,.$

Therefore each ω_p is killed by χ_{sl2} and that finishes the proof. $\square$

Let L be a quasisimple quadratic Lie superalgebra over a field K. Let X be the kernel of the Lie bracket: $\Lambda^2(L) \longrightarrow L$ and Y be the quotient of $S^2(L)$ by the Casimir element Ω of L. So we have exact sequences of L-modules:

$$0 \longrightarrow X \longrightarrow \Lambda^2(L) \longrightarrow L \longrightarrow 0$$
$$0 \longrightarrow K\Omega \longrightarrow S^2(L) \longrightarrow Y \longrightarrow 0$$

Let Ψ_L be the endomorphism of $L^{\otimes 2}$ represented by the diagram:



Since this diagram is symmetric, Ψ_L respects the decomposition: $L^{\otimes 2} = S^2(L) \oplus \Lambda^2(L)$. But Ψ_L respects the exact sequences also and Ψ_L acts on X and Y. If α is a eigenvalue of Ψ_L acting on Y, the corresponding eigenspace will be denoted by Y_α.

Theorem 6.3. *Let L be a quasisimple quadratic Lie superalgebra over a field K which is not sl_2. Let Ω, X, Y and Ψ_L defined as above. Let P be the minimal polynomial of Ψ_L acting on Y.*

Suppose the following conditions hold:

— 6 is invertible in K,

— Ψ_L acts bijectively on Y,

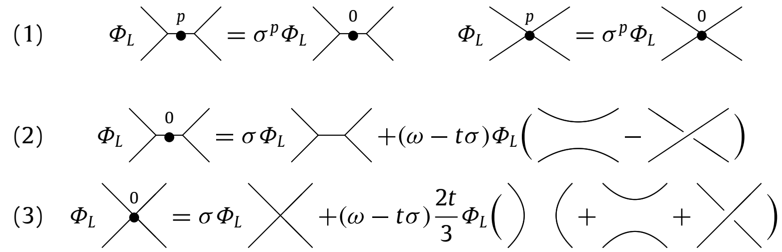
— χ_L is nontrivial on some ω_p or $\partial^\circ P \leq 3$.

Then the degree of P is 2 or 3 and there exist three elements t, σ, ω in K such that:

— $\chi_L(t) = t,\ \forall p \geq 0,\ \chi_L(\omega_p) = \omega\sigma^p$,

— Ψ_L is the multiplication by 0, t and $2t$ on X, $\Lambda^2(L)/X \simeq L$ and $K\Omega$,

— for every $p \geq 0$ we have the following:



If P is of degree 2 (exceptional case), P has 2 nonzero roots α and β in some algebraic extension of K and we have:

$$t = 3(\alpha+\beta), \qquad \sigma = (4\alpha+5\beta)(4\beta+5\alpha), \qquad \omega = 5(\alpha+\beta)(3\alpha+4\beta)(3\beta+4\alpha)$$

$$\mathrm{sdim}(L) = -2\frac{(5\alpha+6\beta)(5\beta+6\alpha)}{\alpha\beta}$$

$$\mathrm{sdim}(X) = 5\frac{(4\alpha+\beta)(4\beta+\alpha)(5\alpha+6\beta)(5\beta+6\alpha)}{\alpha^2\beta^2}$$

$$\alpha \neq \beta \implies \mathrm{sdim}(Y_\alpha) = -90\frac{(\alpha+\beta)^2(6\alpha+5\beta)(3\alpha+4\beta)}{\alpha^2\beta(\alpha-\beta)}.$$

If P is of degree 3 (regular case), P has 3 nonzero roots α, β, γ in some algebraic extension of K and we have:

$$t = \alpha+\beta+\gamma, \qquad \sigma = \alpha\beta+\beta\gamma+\gamma\alpha+2t^2, \qquad \omega = (t+\alpha)(t+\beta)(t+\gamma)$$

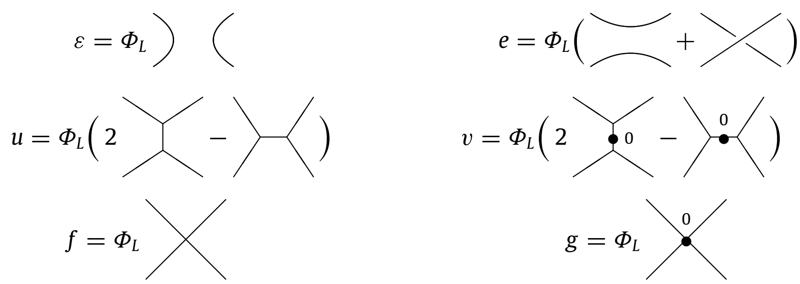
$$\mathrm{sdim}(L) = -\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma}$$

$$\mathrm{sdim}(X) = \frac{\omega(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha^2\beta^2\gamma^2}$$

$$\alpha \neq \beta,\gamma \implies \mathrm{sdim}(Y_\alpha) = \frac{t(2t-\beta)(2t-\gamma)(t+\beta)(t+\gamma)(2t-3\alpha)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}.$$

Remark. In the exceptional case, we may add formally a new root $\gamma = 2t/3$ and a trivial corresponding eigenspace Y_γ. Then the formulas of the superdimensions are exactly the same in the exceptional case or the regular case except that γ is possibly equal to 0.

Proof. Set: $\omega = \chi_L(\omega_0)$ and consider the following endomorphisms in $L^{\otimes 2}$:



These endomorphisms act on $S^2(L)$ and act trivially on $\Lambda^2(L)$.

The degree of P:

Suppose $\chi_L(\omega_p) \neq 0$. We have:

$$\chi_L(\omega_p^2) = \chi_L(\omega_0\omega_{2p}) = \omega\chi_L(\omega_{2p}) \neq 0 \implies \omega \neq 0.$$

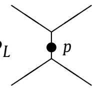
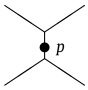
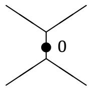
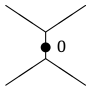
So we can set:

$$\sigma = \frac{\chi_L(\omega_1)}{\omega}$$

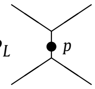
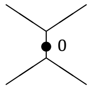
and we have for every $p > 0$:

$$\omega^{p-1}\chi_L(\omega_p) = \chi_L(\omega_0^{p-1}\omega_p) = \chi_L(\omega_1^p) = \omega^p\sigma^p \implies \chi_L(\omega_p) = \sigma^p\omega.$$

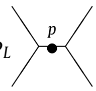
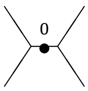
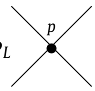
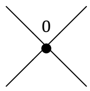
Because of Theorem 5.7, we have also:

$\omega\Phi_L$  $= \Phi_L\omega_0$  $= \Phi_L\omega_p$  $= \omega\sigma^p\Phi_L$ 

and this implies:

Φ_L  $= \sigma^p\Phi_L$ 

Similarly we get for every $p \geq 0$:

Φ_L  $= \sigma^p\Phi_L$  Φ_L  $= \sigma^p\Phi_L$ 

and formulas (1) are proven in this case.

Let E be the vector space formally generated by e, ε, u, v, f and g. Because of Proposition 5.5 the operator Ψ_L induces an action ψ on E defined by:

$$\psi(\varepsilon) = 2t\varepsilon$$
$$\psi(e) = u$$
$$\psi(u) = \frac{t}{3}u + 2f$$
$$\psi(v) = -\omega u + \frac{4t}{3}v + 2g$$
$$\psi(f) = \frac{10t^2}{9}u - \frac{1}{2}v + \frac{2t}{3}f$$
$$\psi(g) = \frac{2t\omega}{3}u + \left(\frac{4t^2}{9} - \frac{\sigma}{2}\right)v + \frac{2t}{3}g.$$

It is easy to see that ψ vanishes on the following element in E:

$$U = g - \sigma f - \frac{t}{3}(v - \sigma u) - t(\omega - t\sigma)e.$$

Since Ψ_L acts bijectively on $S^2(L)/K\Omega$, U induces the trivial endomorphism of $S^2(L)/K\Omega$ and there exists an element $\lambda \in K$ such that the following holds in $\mathrm{End}(L^{\otimes 2})$ (or in $\mathrm{Hom}(L^{\otimes 4}, K)$):

$$g - \sigma f - \frac{t}{3}(v - \sigma u) - t(\omega - t\sigma)e = \lambda\varepsilon.$$

The group $\mathfrak{S}_4$ acts on this equality and the invariant part of it is:

$$g = \sigma f + \left(\frac{2t}{3}(\omega - t\sigma) + \frac{\lambda}{3}\right)(e + \varepsilon).$$

Hence we have also:

$$t(v - \sigma u) = (t(\omega - t\sigma) - \lambda)(2\varepsilon - e).$$

By making a quarter of a turn and composing with the Lie bracket, we get:

$$t(3\omega - 3t\sigma) = 3(t(\omega - t\sigma) - \lambda)$$

which implies: $\lambda = 0$ and we get Formula (3):

$$g = \sigma f + \frac{2t}{3}(\omega - t\sigma)(e + \varepsilon)$$

and also the following:

$$t(v - \sigma u) = t(\omega - t\sigma)(2\varepsilon - e).$$

Let E' be the quotient of E by these two relations. It is easy to see that ψ vanishes on $v - \sigma u - (\omega - t\sigma)(2\varepsilon - e) \in E'$.

For the same reason as above, there is an element $\mu \in K$ such that:

$$v - \sigma u - (\omega - t\sigma)(2\varepsilon - e) = \mu\varepsilon.$$

By making a quarter of a turn and composing with the Lie bracket, we get:

$$3\omega - 3t\sigma - (\omega - t\sigma)3 = \mu.$$

Hence μ is zero and we get the formula (2).

Denote by φ the endomorphism of Y induced by Ψ_L. In this endomorphism algebra we have:

$$\varepsilon = 0 \qquad e = 2$$

$$u = 2\varphi \qquad f = \varphi^2 - \frac{t}{3}\varphi$$

$$v = 2\Big(\frac{20t^2}{9}\varphi - \varphi \circ f + \frac{2t}{3}f\Big) = 2(-\varphi^3 + t\varphi^2 + 2t^2\varphi).$$

The relation $v = \sigma u + (\omega - t\sigma)(2\varepsilon - e)$ implies:

$$2(-\varphi^3 + t\varphi^2 + 2t^2\varphi) = 2\sigma\varphi - 2(\omega - t\sigma)$$

and then:

$$\varphi^3 - t\varphi^2 + (\sigma - 2t^2)\varphi - (\omega - t\sigma) = 0.$$

The minimal polynomial P of φ is then a divisor of the polynomial $Q(X) = X^3 - tX^2 + (\sigma - 2t^2)X - (\omega - t\sigma)$. Since L is quasisimple Y is nonzero and the degree of P is 1, 2 or 3.

Therefore in any case the degree of P is 1, 2 or 3.

Suppose $\partial^\circ P = 1$. Let α be the root of P. Then the endomorphism $v - \alpha e$ of $L^{\otimes 2}$ has its image contained in $K\Omega$ and there is some $\lambda \in K$ such that the following holds in $\mathrm{End}(L^{\otimes 2})$ (or in $\mathrm{Hom}(L^{\otimes 4}, K)$):

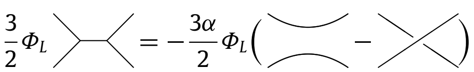
$$v = \alpha e + \lambda\varepsilon.$$

The group $\mathfrak{S}_4$ acts on this equality and the invariant part of this equality is:

$$0 = \left(\frac{2\alpha}{3} + \frac{\lambda}{3}\right)(e + \varepsilon).$$

Then we get: $\lambda = -2\alpha$.

By making a quarter of a turn and composing with the projection: $L^{\otimes 2} \longrightarrow \Lambda^2(L) \subset L^{\otimes 2}$ we get the equality:



Since Ψ_L acts bijectively on Y, α is not zero and $\Lambda^2(L)$ is contained in the image of the cobracket. Therefore the Lie bracket is bijective from $\Lambda^2(L)$ to L. But that is impossible because L is not isomorphic to sl_2. Therefore the degree of P is 2 or 3.

The exceptional case:

Suppose: $\partial^\circ P = 2$ and denote by α and β the roots of P. Since Ψ_L acts bijectively on Y, α and β are not zero. The endomorphism $(v - \alpha e)(v - \beta e)$ is trivial on Y and $\Lambda^2(L)$. Then its image is contained in $K\Omega$ and there exists $\mu \in K$ such that:

$$(v - \alpha e)(v - \beta e) = \mu\varepsilon.$$

So we get:

$$4f + 2\left(\frac{t}{3} - \alpha - \beta\right)v + 2\alpha\beta e = \mu\varepsilon.$$

By taking the invariant part of this equation (under $\mathfrak{S}_4$) we get:

$$4f + \frac{4\alpha\beta}{3}(e + \varepsilon) = \frac{h}{3}(e + \varepsilon)$$

and then:

$$2\left(\frac{t}{3}-\alpha-\beta\right)v=\frac{2\alpha\beta+\mu}{3}(2\varepsilon-e).$$

Since L is not sl_2, v and $2\varepsilon-e$ are linearly independent and we get:

$$t=3(\alpha+\beta)\qquad \mu=-2\alpha\beta$$
$$f=-\frac{\alpha\beta}{2}(e+\varepsilon).$$

By applying Ψ_L to this equality we get:

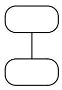
$$-\frac{\alpha\beta}{2}(u+2t\varepsilon)=-\frac{\alpha\beta t}{3}(e+\varepsilon)+\frac{10t^2}{9}u-\frac{v}{2}$$
$$\implies\ v=(4\alpha+5\beta)(4\beta+5\alpha)u+2\alpha\beta(\alpha+\beta)(2\varepsilon-e)$$

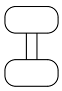
and that implies in any case the formula (2) with: $\sigma=(4\alpha+5\beta)(4\beta+5\alpha)$ and $\omega=5(\alpha+\beta)(3\alpha+4\beta)(3\beta+4\alpha)$. If $\omega=0$ we still have: $\chi_L(\omega_p)=\omega\sigma^p$ and formulas (1) and (3) are consequences of (2).

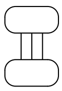
Let d be the superdimension of L and τ be the supertrace operator. Since Ψ_L acts by multiplication by 0, t and $2t$ on X, L and $K\Omega$, we have:

$$\tau(\varphi^0)=\frac{d(d+1)}{2}-1=\frac{(d-1)(d+2)}{2}$$
$$\tau(\Psi_L)=td+2t+\tau(\varphi)$$
$$\tau(\Psi_L^2)=t^2d+4t^2+\tau(\varphi^2)$$
$$\tau(\Psi_L^3)=t^3d+8t^3+\tau(\varphi^3).$$

Using a simple graphical calculus, we get:

$\tau(\Psi_L)=\Phi_L$  $=0$

$\tau(\Psi_L^2)=\Phi_L$  $=4t^2d$

$\tau(\Psi_L^3)=\Phi_L$  $=2t^3d.$

Hence we have:

$$\tau(\varphi^0)=\frac{(d-1)(d+2)}{2}$$
$$\tau(\varphi)=-t(d+2)$$
$$\tau(\varphi^2)=t^2(3d-4)$$
$$\tau(\varphi^3)=t^3(d-8).$$

Since φ has α and β as eigenvalues, we get:

$$t^2(3d-4)+t(\alpha+\beta)(d+2)+\frac{(d-1)(d+2)}{2}\alpha\beta=0$$
$$t^3(d-8)-(\alpha+\beta)t^2(3d-4)-t(d+2)\alpha\beta=0$$

and that implies the following:

$$(\alpha+\beta)(60(\alpha+\beta)^2+(d+2)\alpha\beta)=0$$
$$(d-1)(60(\alpha+\beta)^2+(d+2)\alpha\beta)=0.$$

Suppose $t=0$. Since $(\Psi_L-\alpha)(\Psi_L-\beta)$ vanishes on Y, there exists $\mu\in K$ such that:

$$(\Psi_L-\alpha)(u-\beta e)=2\mu\varepsilon$$
$$\implies\ \left(\frac{t}{3}-\alpha-\beta\right)u+2f=\alpha\beta e+2\mu\varepsilon.$$

Since the left hand side of this equation is invariant under $\mathfrak{S}_4$, it is the same for the other side and we get:

$$2f = \alpha\beta(e + \varepsilon).$$

By composing with the inner product, we get: $d + 2 = 0$. Therefore $d - 1$ is nonzero and we have in any case:

$$60(\alpha + \beta)^2 + (d + 2)\alpha\beta = 0.$$

Then it is not difficult to compute the superdimensions of L and X and we get the desired formula.

Suppose $\alpha \neq \beta$. Denote by d_α and d_β the superdimensions of eigenspaces Y_α and Y_β. We have:

$$d_\alpha + d_\beta = \frac{(d-1)(d+2)}{2}$$
$$\alpha d_\alpha + \beta d_\beta = -t(d + 2)$$

and d_α and d_β can be computed easily.

The regular case:

Consider now the regular case: P is of degree 3 and has 3 nonzero roots α, β, γ.

Since $(\Psi_L - \alpha)(\Psi_L - \beta)(\Psi_L - \gamma)$ acts trivially on Y, there exists $\mu \in K$ such that:

$$(\Psi_L - \alpha)(\Psi_L - \beta)(u - \gamma e) = 2\mu\varepsilon.$$

After reduction we get:

$$\left(\frac{7t^2}{3} - \frac{t}{3}(\alpha + \beta + \gamma) + \alpha\beta + \beta\gamma + \gamma\alpha\right) u + 2(t - \alpha - \beta - \gamma)f - v = \alpha\beta\gamma e + 2\mu\varepsilon.$$

The invariant part of this formula is:

$$2(t - \alpha - \beta - \gamma)f = \frac{2}{3}(\alpha\beta\gamma + \mu)(e + \varepsilon).$$

Since the minimal polynomial of φ has degree 3, f is not a multiple of $e + \varepsilon$. Hence we get:

$$\alpha + \beta + \gamma = t \qquad \mu = -\alpha\beta\gamma$$

and also:

$$(2t^2 + \alpha\beta + \beta\gamma + \gamma\alpha)u - v = \alpha\beta\gamma(e - 2\varepsilon).$$

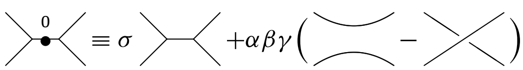
If ω is not zero, P is equal to Q and we have:

$$\alpha\beta + \beta\gamma + \gamma\alpha = \sigma - 2t^2 \qquad \alpha\beta\gamma = \omega - t\sigma.$$

Otherwise we can set: $\sigma = \alpha\beta + \beta\gamma + \gamma\alpha + 2t^2$ and we have:

$$v = \sigma u + \alpha\beta\gamma(2\varepsilon - e)$$

and then:



By applying the Lie bracket, we get: $0 = 2\omega = 2t\sigma + 2\alpha\beta\gamma$. In this case we have: $\alpha\beta\gamma = \omega - t\sigma$ and the formula (2) follows. As above formulas (1) and (3) are easy to check.

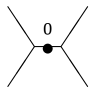
In any case t, σ, ω can be expressed in term of α, β, γ. As above we get the following:

$$\tau(\varphi^0) = \frac{(d-1)(d+2)}{2}$$
$$\tau(\varphi) = -t(d + 2)$$
$$\tau(\varphi^2) = t^2(3d - 4)$$
$$\tau(\varphi^3) = t^3(d - 8).$$

Since φ has α, β, γ as eigenvalues, we get:

$$t^3(d - 8) - t^3(3d - 4) - t(\sigma - 2t^2)(d + 2) - \frac{(d-1)(d+2)}{2}\alpha\beta\gamma = 0$$
$$\Longrightarrow \quad (d + 2)(\alpha\beta\gamma d + (2t - \alpha)(2t - \beta)(2t - \gamma)) = 0.$$

Let F be the endomorphism of $L^{\otimes 2}$ represented by the diagram:



Because of the formula (2), F acts by 2ω on L and by $2(\omega - t\sigma)$ on X. It is trivial on $S^2(L)$. Therefore we get:

$$0 = \tau(F) = 2\omega d + 2(\omega - t\sigma)\frac{d(d-3)}{2} \quad\Longrightarrow\quad \omega d(d-1) = t\sigma d(d-3).$$

Suppose $d = -2$. Then we have:

$$3\omega = 5t\sigma$$

and this implies:

$$-\frac{(2t-\alpha)(2t-\beta)(2t-\gamma)}{\alpha\beta\gamma} = -\frac{4t^3 + 2t(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma}{\alpha\beta\gamma}$$
$$= -2 + \frac{3\alpha\beta\gamma - 2t\sigma}{\alpha\beta\gamma} = -2 + \frac{3\omega - 5t\sigma}{\alpha\beta\gamma} = -2.$$

Therefore in any case we have:

$$\alpha\beta\gamma d + (2t-\alpha)(2t-\beta)(2t-\gamma) = 0$$

and d and the superdimension of X are easy to compute.

If α is different from β and γ, we have the following (with d_α =sdimY_α):

$$(\alpha-\beta)(\alpha-\gamma)d_\alpha = \tau(\varphi^2 - (\beta+\gamma)\varphi + \beta\gamma\varphi^0)$$
$$= t^2(3d-4) + t(d+2)(\beta+\gamma) + \frac{(d-1)(d+2)}{2}\beta\gamma$$

and that gives the value of d_α. □

7. The eight characters

7.1. The gl case

Let E be a supermodule of superdimension m. Take a homogeneous basis $\{e_i\}$ of E and denote by $\{e_{ij}\}$ the corresponding basis of $gl(E)$. Let $sl(E) \subset gl(E)$ be the Lie superalgebra of endomorphisms of E with zero supertrace. The map sending $\alpha \otimes \beta \in gl(E) \otimes gl(E)$ to the supertrace of $\alpha \circ \beta$ is an nonsingular invariant bilinear form on $gl(E)$ and $gl(E)$ is a quadratic Lie superalgebra. If m is invertible, $sl(E)$ is also a quadratic Lie superalgebra. If $m = 0$, the inner form is singular on $sl(E)$, but the quotient of $sl(E)$ by its center is a quadratic Lie superalgebra $psl(E)$.

If the coefficient ring is a field K, we have the following:

— suppose m is invertible in K and $\dim(E) > 1$. Then $sl(E)$ is quasisimple and the character $\chi_{sl(E)}$ is well defined.

— suppose $m = 0$ and $\dim(E) > 2$. Then $psl(E)$ is quasisimple and the character $\chi_{psl(E)}$ is well defined.

Theorem 7.2. *Let $\mathbf{Z}[t,u]$ be the polynomial algebra generated by variables t and u of degree 1 and 2 respectively. For each $m \in \mathbf{Z}$, denote by γ_m the ring homomorphism sending t to m and u to 1. Then there exists a unique graded algebra homomorphism χ_{gl} from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[t,u]$ such that the following hold for every super vector space E of superdimension m over a field K:*

— for every closed oriented curve Γ colored by $gl(E)$-representations, and every finite set X, we have:

$$\forall\alpha \in \Lambda_k,\ \forall u \in \mathcal{A}_{\mathbf{Z}}(\Gamma, X), \quad \Phi_{gl(E)}(\alpha u) = \gamma_m \circ \chi_{gl}(\alpha)\Phi_{gl(E)}(u)$$

— if m is invertible in K and $\dim(E) > 1$, $\chi_{sl(E)}$ is the composite $\gamma_m \circ \chi_{gl}$

— if $m = 0$ and $\dim(E) > 2$, $\chi_{psl(E)}$ is the composite $\gamma_0 \circ \chi_{gl}$.

Moreover χ_{gl} satisfies the following:

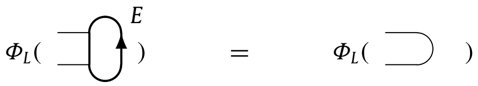
$$\chi_{gl}(t) = t \quad \text{and} \quad \forall p \geq 0, \qquad \chi_{gl}(\omega_p) = \omega\sigma^p$$

with: $\omega = 2t(t^2 - 4u)$ and $\sigma = 2(t^2 - 2u)$.

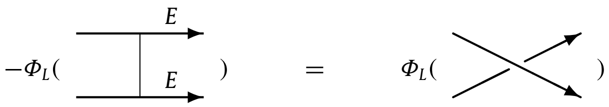
Proof. Let E be a finite-dimensional free $\mathbf{Z}$-supermodule of superdimension m. Let $\{e_i\}$ be a homogeneous basis of E and $\{e_{ij}\}$ be the corresponding basis of $L = gl(E)$. The Casimir element of L is:

$$\Omega = \sum_{ij} (-1)^{\partial^\circ e_j} e_{ij} \otimes e_{ji}.$$

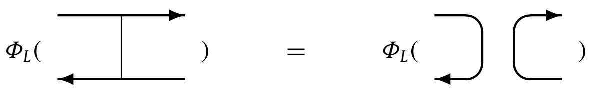
Since the inner product of x and y in L is $\langle x, y\rangle = \tau_E(xy)$ we have the following:



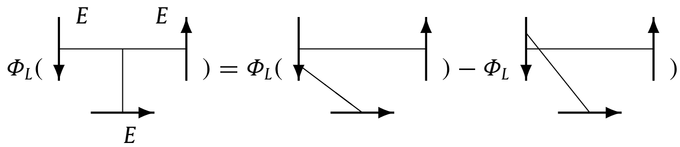
Moreover, it is not difficult to show the following:

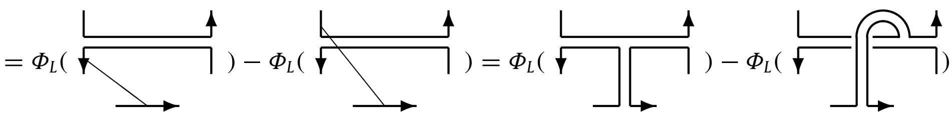


Whence:



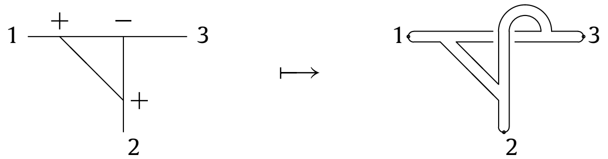
and we get:





Therefore, to compute the image by Φ_L of a $(\emptyset, [n])$-diagram D, we may proceed as follows:

Let $S(D)$ be the set of functions α from the set of 3-valent vertices of D to $\{\pm 1\}$. For every $\alpha \in S(D)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(x)$. If $\alpha \in S(D)$ is given we may construct a thickening of D by using the given cyclic ordering of edges ending at a 3-valent vertex x if $\alpha(x) = 1$ and the other one if not, and we get an oriented surface $\Sigma_\alpha(D)$ equipped with n numbered points in its boundary.



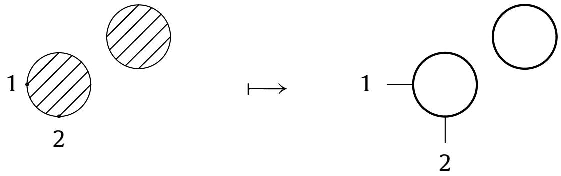
Denote by S_n the set of isomorphism classes of oriented connected surfaces equipped with n numbered points in its boundary. Under the connected sum, $S = S_0$ is a monoid and acts on S_n. This monoid is a graded commutative monoid freely generated by the disk D of degree 1 and the torus T of degree 2. The set S_n is a graded S-set with dim H_1 as degree. Let $\mathbf{Z}[S_n]$ and $\mathbf{Z}[S]$ be the free modules generated by S_n and S. They are graded modules, and $\mathbf{Z}[S]$ is a polynomial algebra acting on $\mathbf{Z}[S_n]$.

If D is connected, the sum

$$s(D) = \sum_\alpha \varepsilon(\alpha)\Sigma_\alpha(D)$$

lies in $\mathbf{Z}[S_n]$. It is easy to check that s is compatible with AS and IHX relations and induces a well-defined graded homomorphism from $F_{\mathbf{Z}}(n)$ to $\mathbf{Z}[S_n]$. Moreover, this homomorphism is $\Lambda_{\mathbf{Z}}[S]$-linear with respect to a character χ from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[S] = \mathbf{Z}[D, T]$.

On the other hand, for each $\Sigma \in S_n$ we have a diagram $\partial(\Sigma)$ in $\mathcal{D}(\Gamma, [n])$ where Γ is colored by E: $\partial(\Sigma)$ is the boundary of Σ colored by E with intervals added near each marked point:



We can extend ∂ linearly and for every $\Sigma \in \mathbf{Z}[S_n]$, $\Phi_L(\partial(\Sigma))$ is well defined in $L^{\otimes n}$. Moreover we have:

$$\Phi_L(D) = \sum_\alpha \varepsilon(\alpha)\Phi_L(\partial \Sigma_\alpha(D)) = \Phi_L(\partial s(D)).$$

Hence for $a \in \Lambda_{\mathbf{Z}}$, we have:

$$\begin{aligned}\Phi_L(aD) &= \Phi_L(\partial s(aD)) = \Phi_L(\partial \chi(a) s(D)) = \Phi_L(\chi(a)\partial s(D))\\ &= \gamma_m(\chi(a))\Phi_L(\partial s(D)) = \gamma_m(\chi(a))\Phi_L(D)\end{aligned}$$

and the first part of the theorem is proven in the case $\Gamma = \emptyset$ (with $\chi_{gl} = \chi$). The general case follows.

Suppose now E is a super vector space of dimension > 1 over a field K.

If m is invertible in K, $sl(E)$ is quasisimple and $gl(E)$ is semisimple: $gl(E) = sl(E) \oplus K$. Since Φ_K is trivial, we have:

$$\begin{aligned}\Phi_{sl(E)}(aD) = \Phi_{gl(E)}(aD) &= \gamma_m(\chi_{gl}(a))\Phi_{gl(E)}(D) = \gamma_m(\chi_{gl}(a))\Phi_{sl(E)}(D)\\ &\Longrightarrow \chi_{sl(E)} = \gamma_m \circ \chi_{gl}.\end{aligned}$$

Suppose now: $m = 0$ and $\dim(E) > 2$. In this case $psl(E)$ is quasisimple. Since the Lie bracket on $gl(E)$ takes values in $sl(E)$, $\Phi_{gl(E)}(D)$ lies in $sl(E)^{\otimes n}$ and for every $a \in \Lambda_{\mathbf{Z}}$ the equality

$$\Phi_{gl(E)}(aD) = \gamma_m(\chi_{gl}(a))\Phi_{gl(E)}(D)$$

holds in $sl(E)^{\otimes n}$. Hence in the quotient $psl(E)^{\otimes n}$ we have:

$$\Phi_{psl(E)}(aD) = \gamma_m(\chi_{gl}(a))\Phi_{psl(E)}(D)$$

and we get:

$$\chi_{psl(E)}) = \gamma_0 \circ \chi_{gl}.$$

In order to prove the last part of the theorem, it is enough to determine $\chi_{sl(E)}(\omega_p)$ for $K = \mathbf{Q}$ and for infinitely many values of m. Suppose now $m > 2$ and E has no odd part. Then $L = sl(E)$ is the classical Lie algebra sl_m. The morphism $\Psi = \Psi_L$ from $L^{\otimes 2}$ to itself is the morphism:

$$x \otimes y \mapsto \sum_{ij} [x, e_{ij}] \otimes [e_{ji}, y]$$

and because of Theorem 6.3 we have to determine eigenvalues of Ψ acting on $Y = S^2(L)/\Omega$.

Denote by τ the trace operator. Let $f : L^{\otimes 2} \longrightarrow L$ be the following morphism:

$$f : x \otimes y \mapsto xy + yx - \frac{2}{m}\tau(xy)\mathrm{Id}.$$

Since $m > 2$, f is surjective and L-linear. We have:

$$\begin{aligned}f\Psi(x \otimes y) &= \sum_{ij}((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji}) + (e_{ji}y - ye_{ji})(xe_{ij} - e_{ij}x)) - \sum_{ij}\frac{2}{m}\tau((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji}))\\ &= mxy + \tau(xy) + myx + \tau(yx) - \frac{2}{m}\tau(mxy + \tau(xy)) = mxy + myx - 2\tau(xy) = mf(x \otimes y).\end{aligned}$$

The map f factorizes through Y and there is an exact sequence:

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow L \longrightarrow 0$$

compatible with the action of Ψ and Ψ induces the multiplication by m on L.

The module Z can be seen as a submodule of $L^{\otimes 2}$ and the morphisms sending $x \otimes y$ to xy, yx, $x \otimes y - y \otimes x$ are trivial on Z. If z lies in L, denote by z_{ij} the entries of z. We have:

$$\begin{aligned}\Psi(x \otimes y) &= \sum (x_{ki}e_{kj} - x_{jk}e_{ik}) \otimes (y_{il}e_{jl} - y_{lj}e_{li}) \\ &= \sum (xy)_{kl}e_{kj} \otimes e_{jl} + (yx)_{lk}e_{ik} \otimes e_{li} - x_{ki}e_{kj} \otimes y_{lj}e_{li} - y_{il}e_{ik} \otimes x_{jk}e_{jl}.\end{aligned}$$

Therefore the morphism Ψ is equal on Z to the morphism Ψ' defined by:

$$\Psi'(x \otimes y) = -2 \sum xe_{ij} \otimes ye_{ji}$$

and we have:

$$\Psi'^2(x \otimes y) = 4 \sum xe_{ij}e_{kl} \otimes ye_{ji}e_{lk} = 4x \otimes y.$$

Therefore the minimal polynomial of Ψ acting on Y is of degree three with roots m, 2, -2. Then Theorem 6.3 implies the following:

$$\chi_L(t) = m \quad \forall p \geq 0, \qquad \chi_L(\omega_p) = 2m(m+2)(m-2)(2m^2-4)^p$$

and that finishes the proof. $\square$

7.3. The osp case

Let E be a supermodule of superdimension m equipped with a supersymmetric nonsingular bilinear form $\langle\ ,\ \rangle$ of degree zero. We will say that E is a quadratic supermodule. For every endomorphism α of E, we have a endomorphism α^* defined by:

$$\forall x, y \in E \qquad \langle \alpha^*(x), y\rangle = (-1)^{pq}\langle x, \alpha(y)\rangle$$

where p is the degree of x and q is the degree of α. An endomorphism α is *antisymmetric* if $\alpha^* = -\alpha$. Let $L = osp(E)$ be the Lie superalgebra of antisymmetric endomorphisms of E. The superdimension of L is $d = m(m-1)/2$. With the same notation as before, a Casimir element of L is:

$$\Omega = \frac{1}{2} \sum_{i,j} (-1)^{\partial^\circ e_j} (e_{ij} - e_{ij}^*) \otimes (e_{ji} - e_{ji}^*)$$

and with this Casimir element, $t = m - 2$. The bilinear form corresponding to Ω is half the supertrace of the product and L is a quadratic Lie superalgebra.

If $\dim(E) = \mathrm{sdim}(E) < 3$, L is abelian. Otherwise L is quasisimple.

Theorem 7.4. *Let $\mathbf{Z}[t, v]$ be the polynomial algebra generated by variables t and v of degree 1. Then there exists a unique graded algebra homomorphism χ_{osp} from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[t, v]$ such that:*

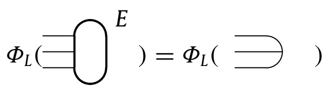
– for every quadratic super vector space E with $\dim(E) > 2$ or $\mathrm{sdim}(E) = -2$, $\chi_{osp(E)}$ is the composite $\gamma \circ \chi_{osp}$, where γ is the ring homomorphism sending t to $\mathrm{sdim}(E) - 2$ and v to 1.

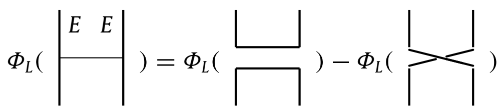
Moreover χ_{osp} satisfies the following:

$$\chi_{osp}(t) = t \quad \text{and} \quad \forall p \geq 0, \quad \chi_{osp}(\omega_p) = \omega\sigma^p$$

with: $\omega = 2(t - v)(t - 2v)(t + 4v)$ and $\sigma = 2(t - 2v)(t + 3v)$.

Proof. Let E be a quadratic super vector space and L be the Lie superalgebra $osp(E)$. Let D be a L-colored diagram. If we change the orientation of a component colored by E, $\Phi_L(D)$ is unchanged. Therefore we may consider in D unoriented components colored by E. On the other hand it is easy to see the following:

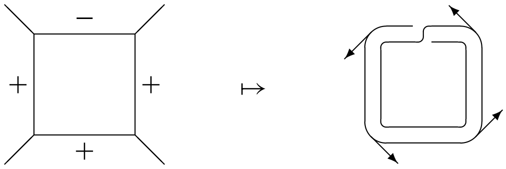




Therefore, in order to compute the image under Φ_L of a $(\emptyset, [n])$-diagram D, we may proceed as follows:

Let $S(D)$ be the set of functions from the set of edges of D having no 1-valent boundary point to the set $\{\pm 1\}$. For every $\alpha \in S(D)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(a)$. If $\alpha \in S(D)$ is given we may construct a thickening of D by using the given cyclic ordering of edges ending at each 3-valent vertex and making a half-twist near every edge a with negative $\alpha(a)$.

So we get an unoriented surface $\Sigma_\alpha(D)$ equipped with n numbered points in its boundary and a local orientation of $\partial\Sigma_\alpha(D)$ at each of these points.



Denote by US_n the set of isomorphism classes of connected surfaces Σ equipped with n numbered points in its boundary and an orientation of $\partial\Sigma$ at each of these points. Under the connected sum, $US = US_0$ is a monoid and acts on US_n. This monoid is a graded commutative monoid generated by the disk D, the projective plane P and the torus T and the only relation is: $PT = P^3$.

Let $\mathbf{Z}(US_n)$ be the $\mathbf{Z}$-module generated by the elements of US_n with the following relations:

If Σ' is obtained from Σ by changing the local orientation near one point, $\Sigma + \Sigma'$ is trivial in $\mathbf{Z}(US_n)$.

Then $\mathbf{Z}[US]$ is a commutative algebra and $\mathbf{Z}(US_n)$ is a graded $\mathbf{Z}[US]$-module.

If D is connected, the sum $s(D) = \sum_\alpha \varepsilon(\alpha)\Sigma_\alpha(D)$ lies in $\mathbf{Z}[US_n]$. It is easy to check that s is compatible with AS and IHX relations and induces a well-defined graded homomorphism from $F_{\mathbf{Z}}(n)$ to $\mathbf{Z}[US_n]$. Moreover this homomorphism is $\Lambda_{\mathbf{Z}}[US]$-linear with respect to a character χ from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[US] = \mathbf{Z}[D, P, T]/(PT - P^3)$.

On the other hand, we have a map ∂ from US_n and $\mathbf{Z}(US_n)$ to $F_{\mathbf{Z}}(n)$ by sending each surface Σ with numbered points in $\partial\Sigma$ to the boundary $\partial\Sigma$ colored by E with intervals added near each marked point. If D is a diagram, $\Phi_L(D)$ is equal to the sum $\sum_\alpha \varepsilon(\alpha)\Phi_L(\partial\Sigma_\alpha(D)) = \Phi_L(\partial s(D))$. Therefore if u is an element of $\Lambda_{\mathbf{Z}}$, we have $\chi_L(u) = \chi_L(\partial\chi(u))$. Since $\chi_L \circ \partial$ is a ring homomorphism sending D to $m = \operatorname{sdim} E$ and P and T to 1, the character χ_L factorizes through $\mathbf{Z}[D, P] = \mathbf{Z}[US]/(T - P^2)$ and the first part of the theorem is proven (with $t = D - 2P$, $v = P$).

In order to prove the last part of the theorem, it is enough to consider the case where E is a classical vector space over $\mathbf{Q}$ of large dimension m. Then the second symmetric power $S^2(L)$ decomposes into four simple L-modules E_0, E_1, E_2, E_3 of dimensions 1, $(m-1)(m+2)/2$, $m(m-1)(m-2)(m-3)/4!$, $m(m+1)(m+2)(m-3)/12$. Therefore we have the decomposition: $Y = E_1 \oplus E_2 \oplus E_3$. Moreover the Casimir homomorphism acts on E_1, E_2, E_3 by multiplication by $2m, 4m - 16, 4m - 4$. On the other hand, this homomorphism is equal to $4t - 2\Psi_L$. Therefore Ψ_L acts on E_1, E_2, E_3 by multiplication by $m - 4, 4, -2$.

The last part of the proof is an straightforward consequence of Theorem 6.3. □

Remark. The use of surfaces in the *gl*- and *osp*-cases was introduced in a slightly different way by Bar-Natan to produce weight functions [1].

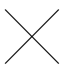
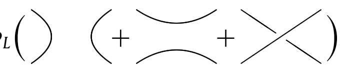
7.5. The exceptional case

Consider a quasisimple quadratic Lie superalgebra L over a field K of characteristic 0. This Lie superalgebra L is said to be exceptional if it satisfies the following condition:

— the square of the Casimir generates in degree 4 the center of the enveloping algebra $\mathcal{U}$ of L.

Exceptional Lie algebras E_6, E_7, E_8, F_4, G_2 satisfy this property. But it is also the case for sl_2, sl_3, $osp(E)$ with $\operatorname{sdim}(E) = 2$ or 8, $psl(E)$ with $\operatorname{sdim}(E) = 0$ and the exceptional Lie superalgebras $G(3)$ and $F(4)$.

Consider the following elements in $F_K(4)$:

$u = \Phi_L$  $v = \Phi_L$ 

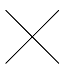
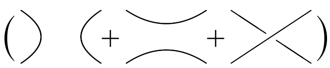
These elements are invariants elements in $S^4(L)$. But the condition satisfied by L implies that the invariant part of $S^4(L)$ is generated by v. Therefore u is a multiple of v and the homomorphism Ψ_L has only two eigenvalues on $S^2(L)/\Omega$. Hence we may apply Theorem 6.3 in the exceptional case and we get:

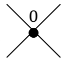
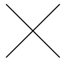
Theorem 7.6. *Let L be an exceptional quasisimple quadratic Lie superalgebra over a field K of characteristic zero. Then there exist σ and ω in K and two elements α and β in some extension of K such that:*

$$t = 3(\alpha+\beta) \qquad \sigma = (4\alpha+5\beta)(5\alpha+4\beta) \qquad \omega = 5(\alpha+\beta)(3\alpha+4\beta)(4\alpha+3\beta)$$

$$\chi_L(t) = t \qquad \forall p \geq 0, \quad \chi_L(\omega_p) = \omega\sigma^p$$

$$\operatorname{sdim}(L) = -2\frac{(5\alpha+6\beta)(6\alpha+5\beta)}{\alpha\beta}$$

Φ_L  $= -\frac{\alpha\beta}{2}\Phi_L$ 

Φ_L  $= (3\alpha + 4\beta)(4\alpha + 3\beta)\Phi_L$ 

Remark. In this theorem, we may consider the Casimir Ω, and then α and β up to a scalar. So α and β may be considered as degree 1 variables related by some linear relation.

Case by case we get the following:

L	sdim(L)	α/β	σ	ω
E_6	78	-3	$\frac{77}{36}t^2$	$\frac{25}{12}t^3$
E_7	133	-4	$\frac{176}{81}t^2$	$\frac{520}{243}t^3$
E_8	248	-6	$\frac{494}{225}t^2$	$\frac{98}{45}t^3$
F_4	52	$-5/2$	$\frac{170}{81}t^2$	$\frac{480}{243}t^3$
G_2	14	$-5/3$	$\frac{65}{36}t^2$	$\frac{55}{36}t^3$
$sl_2, G(3)$	3	$-4/3$	$\frac{8}{9}t^2$	0
$sl_3, F(4)$	8	$-3/2$	$\frac{14}{9}t^2$	$\frac{10}{9}t^3$
$osp(8)$	28	-2	$2t^2$	$\frac{50}{27}t^3$
$osp(2)$	1	$-5/4$	0	$-\frac{40}{27}t^3$
$psl(E)$	-2	-1		0

In this table, $osp(n)$ means any quasisimple Lie superalgebra $osp(E)$ where E is a quadratic supermodule with $\mathrm{sdim}(E) = n$.

In the case sl_2 or $G(3)$ or $psl(E)$ (with $\mathrm{sdim}(E) = 0$), the induced character kills every ω_p and the value of σ is useless. In the case $psl(E)$, the character is determined by any nonabelian $gl(F)$. Then $\chi_{psl(E)}$ is determined by $gl(1|1)$. But every double bracket $[[x, y], z]$ vanishes in this Lie superalgebra. Therefore $\Phi_{gl(1|1)}$ is trivial on Λ in positive degree and the character $\chi_{psl(E)}$ is the trivial character.

Remark. The characters $\chi_{G(3)}$ and $\chi_{F(4)}$ are equal to χ_{sl2} and χ_{sl3} on the algebra generated by t and the ω_p's. These characters are actually equal to χ_{sl2} and χ_{sl3} on Λ. This result was proven by Patureau-Mirand [18].

Conjecture. *Let R be the subalgebra $\mathbf{Q}[\alpha + \beta, \alpha\beta]$ of $\mathbf{Q}[\alpha, \beta]$ where α and β are two formal parameters of degree 1. Then there exists a unique graded algebra homomorphism χ_{exc} from Λ to R such that:*

$$\chi_{exc}(t) = 3(\alpha + \beta)$$
$$\forall p \geq 0, \quad \chi_{exc}(\omega_p) = 5(\alpha + \beta)(3\alpha + 4\beta)(4\alpha + 3\beta)(4\alpha + 5\beta)^p(5\alpha + 4\beta)^p.$$

Remark. This conjecture is actually equivalent to a conjecture of Deligne [6]. If Deligne's conjecture is true, there exists a monoidal category $\mathcal{C}$ which is linear over an algebra $\mathbf{Q}(\lambda)$ and looks like the category of representations of some virtual exceptional Lie algebra. It is not difficult to construct a functor from the category Δ to $\mathcal{C}$ and we get an algebra homomorphism from Λ to the coefficient algebra $\mathbf{Q}(\lambda)$. But this morphism is equivalent to a graded homomorphism χ from Λ to R and the desired properties of χ are easy to check.

Conversely if such a morphism χ exists, we get an algebra homomorphism χ' from $\Lambda[d]$ to the localized algebra $R' = R[\frac{1}{\alpha\beta}]$ by:

$$\chi'(d) = -2\frac{(5\alpha+6\beta)(6\alpha+5\beta)}{\alpha\beta}.$$

Then we may force $\Lambda[d]$ to act on morphisms in the category Δ (and not only on special diagrams). So we get a new category Δ_1 which is linear over $\Lambda[d]$, where d represents the circle. By tensoring Δ_1 over $\Lambda[d]$ by R', we get a category Δ_2 which is linear over R'. If we kill every morphism $f : X \longrightarrow Y$ in Δ_2 such that the trace of $f \circ g$ vanishes for every $g : Y \longrightarrow X$, we get a category Δ_3 which satisfies all Deligne properties. Hence we have a positive answer to Deligne's conjecture.

Remark. Suppose the conjecture is true. Let λ be any element in $\Lambda_{\mathbf{Z}}$ and $P = P(\alpha, \beta)$ be its image under χ_{exc}. The expression $P(\alpha, \beta)$ is known if α/β lies in the set $E = \{-3, -4, -6, -5/2, -5/3, -4/3, -3/2, -2, -5/4, -1\}$. Therefore P is well defined modulo the following polynomial:

$$\Pi = (\alpha+\beta)[1, 2][1, 3][1, 4][1, 6][2, 3][2, 5][3, 4][3, 5][4, 5]$$

with: $[p, q] = (p\alpha + q\beta)(p\beta + q\alpha)$.

By looking carefully at each character corresponding to the exceptional Lie algebras we can check that $P(\alpha, \beta)$ is an integer if α/β or β/α lies in E and $\alpha + \beta$ and $\alpha\beta/2$ are integers. So we have a stronger conjecture:

Conjecture. *There exists a unique graded algebra homomorphism χ_{exc} from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[\alpha+\beta, \alpha\beta/2]$ such that:*

$$\chi_{exc}(t) = 3(\alpha+\beta)$$
$$\forall p \geq 0, \quad \chi_{exc}(\omega_p) = 5(\alpha+\beta)(3\alpha+4\beta)(4\alpha+3\beta)(4\alpha+5\beta)^p(5\alpha+4\beta)^p.$$

7.7. *The super case*

There exists an interesting Lie superalgebra depending on a parameter α called $\mathrm{D}(2, 1, \alpha)$. This algebra is simple and has a nonsingular bilinear supersymmetric invariant form and a Casimir element. Therefore it produces a character on Λ depending on the parameter α. Actually this algebra produces a graded character from $\Lambda_{\mathbf{Z}}$ to a polynomial algebra $\mathbf{Z}[\sigma_2, \sigma_3]$.

Consider oriented 2-dimensional free $\mathbf{Z}$-modules E_1, E_2, E_3 and denote by X the module $E_1 \otimes E_2 \otimes E_3$. This module X is a module over the Lie algebra $L' = sl(E_1) \oplus sl(E_2) \oplus sl(E_3)$.

Since E_i is oriented, there is a canonical isomorphism $x \otimes y \mapsto x{\wedge}y$ from $\Lambda^2(E_i)$ to $\mathbf{Z}$. On the other hand, we have a map from $S^2(E_i)$ to $sl(E_i)$ sending $x \otimes y$ to the endomorphism $x.y : z \mapsto x\, y{\wedge}z + y\, x{\wedge}z$.

For each $i \in \{1, 2, 3\}$ take an element $f_i \in sl(E_i)$ which is congruent to the identity mod 2. Let A be the polynomial algebra $\mathbf{Z}[a, b, c]$ divided by the only relation $a + b + c = 0$. Then we can define a Lie superalgebra L over A by the following:

— the even part L_0 of L is the A-submodule of $A[1/2] \otimes (\oplus_i sl(E_i))$ generated by $sl(E_1), sl(E_2), sl(E_3)$ and $(af_1 + bf_2 + cf_3)/2$
— the odd part L_1 of L is the A-module $A \otimes X$
— the Lie bracket on $L_0 \otimes L_0$ is the standard Lie bracket on $sl(E_i) \otimes sl(E_i)$ and vanishes on $sl(E_i) \otimes sl(E_j)$ for $i \neq j$
— the Lie bracket on $L_0 \otimes L_1$ is the standard action of $\oplus_i sl(E_i)$ on X
— the Lie bracket on $L_1 \otimes L_0$ is the opposite of the standard action of $\oplus_i sl(E_i)$ on X
— the Lie bracket on $X \otimes X$ is defined by:

$$[x \otimes y \otimes z, x' \otimes y' \otimes z'] = \frac{1}{2}(a\, x.x'\, y{\wedge}y'\, z{\wedge}z' + b\, x{\wedge}x'\, y.y'\, z{\wedge}z' + c\, x{\wedge}x'\, y{\wedge}y'\, z.z').$$

It is not difficult to see that L is a Lie superalgebra over A with superdimension $9 - 8 = 1$. The Jacobi relation holds because $a + b + c = 0$. If we take a character from A to $\mathbf{C}$, we get a complex Lie superalgebra. Up to isomorphism, this algebra depends only on one parameter α and is called $\mathrm{D}(2, 1, \alpha)$. Here this algebra L will be denoted by $\widetilde{\mathrm{D}}(2, 1)$.

In order to define a Casimir element in $\widetilde{\mathrm{D}}(2, 1)$, we need some notations. Consider for each $i = 1, 2, 3$ a direct basis $\{\varepsilon_{ij}\}$ of E_i and the dual basis $\{\varepsilon'_{ij}\}$ with respect to the form $\wedge$:

$$\forall x \in E_i \quad \sum_j \varepsilon_{ij}(\varepsilon'_{ij}{\wedge}x) = \sum_j (x{\wedge}\varepsilon_{ij})\varepsilon'_{ij} = x.$$

For each i, the trace of the product is an invariant form on $sl(E_i)$, and, corresponding to this form, we have a Casimir type element $\Omega_i = \sum_j e_{ij} \otimes e'_{ij}$. This element belongs to $L \otimes L \otimes \mathbf{Z}[1/2]$, but $2\omega_i$ lies in $L \otimes L$. We have also a Casimir element $\pi \in X \otimes X$ defined by:

$$\pi = \sum_{ijk} (\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}).$$

Lemma 7.7.1. *For each $i \in \{1, 2, 3\}$ and $x \in E_i$, we have the following:*

$$\sum_j \varepsilon_{ij} \otimes x.\varepsilon'_{ij} = 2\sum_j e_{ij}(x) \otimes e'_{ij},$$
$$\sum_j x.\varepsilon_{ij} \otimes \varepsilon'_{ij} = -2\sum_j e_{ij} \otimes e'_{ij}(x).$$

Proof. Denote by τ the trace map. For every $\alpha \in \mathrm{End}(E_i)$ we have:

$$\begin{aligned}\sum_j \varepsilon_{ij}\tau((x.\varepsilon'_{ij})\alpha) &= \sum_j \varepsilon_{ij}(\varepsilon'_{ij}\wedge\alpha(x)) + \sum_j \varepsilon_{ij}(x\wedge\alpha(\varepsilon'_{ij}))\\ &= \alpha(x) - \sum_j \varepsilon_{ij}(\alpha(x)\wedge\varepsilon'_{ij}) = 2\alpha(x) = 2\sum_j e_{ij}(x)\tau(e'_{ij}\alpha)\end{aligned}$$

and that gives the first formula. The second one is obtained in the same way. $\square$

Lemma 7.7.2. *Let K be the fraction field of A. Then $\widetilde{\mathrm{D}}(2, 1) \otimes K$ has an invariant bilinear form and the corresponding Casimir element is:*

$$\Omega = -a\Omega_1 - b\Omega_2 - c\Omega_3 + \pi.$$

Moreover the cobracket induced by Ω sends $\widetilde{\mathrm{D}}(2, 1)$ to $\widetilde{\mathrm{D}}(2, 1) \otimes \widetilde{\mathrm{D}}(2, 1)$.

Proof. Let $x \otimes y \otimes z$ be an element of X. We have:

$$\begin{aligned}x\otimes y\otimes z(\pi) &= \sum_{ijk}[x\otimes y\otimes z, \varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k}]\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k})\\ &\quad -\sum_{ijk}(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes[x\otimes y\otimes z, \varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}]\\ &= \frac{1}{2}(aZ_1 + bZ_2 + cZ_3)\end{aligned}$$

with:

$$\begin{aligned}Z_1 &= \sum_{ijk} x.\varepsilon_{1i}\, y\wedge\varepsilon_{2j}\, z\wedge\varepsilon_{3k}\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}) - \sum_{ijk}(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes x.\varepsilon'_{1i}\, y\wedge\varepsilon'_{2j}\, z\wedge\varepsilon'_{3k}\\ Z_2 &= \sum_{ijk} x\wedge\varepsilon_{1i}\, y.\varepsilon_{2j}\, z\wedge\varepsilon_{3k}\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}) - \sum_{ijk}(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes x\wedge\varepsilon'_{1i}\, y.\varepsilon'_{2j}\, z\wedge\varepsilon'_{3k}\\ Z_3 &= \sum_{ijk} x\wedge\varepsilon_{1i}\, y\wedge\varepsilon_{2j}\, z.\varepsilon_{3k}\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}) - \sum_{ijk}(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes x\wedge\varepsilon'_{1i}\, y\wedge\varepsilon'_{2j}\, z.\varepsilon'_{3k}.\end{aligned}$$

Using Lemma 7.7.1, Z_1 is easy to compute:

$$\begin{aligned}Z_1 &= \sum_{ijk} x.\varepsilon_{1i}\otimes(\varepsilon'_{1i}\otimes y\otimes z) - \sum_{ijk}(\varepsilon_{1i}\otimes y\otimes z)\otimes x.\varepsilon'_{1i}\\ &= -2\sum_{ijk} e_{1i}\otimes(e'_{1i}(x)\otimes y\otimes z) - 2\sum_{ijk}(e_{1i}(x)\otimes y\otimes z)\otimes e'_{1i} = 2x\otimes y\otimes z(\Omega_1)\end{aligned}$$

and similarly we get: $Z_2 = 2x \otimes y \otimes z(\Omega_2)$, $Z_3 = 2x \otimes y \otimes z(\Omega_3)$. Therefore we have:

$$x\otimes y\otimes z(\Omega) = x\otimes y\otimes z(-a\Omega_1 - b\Omega_2 - c\Omega_3) + \frac{1}{2}(aZ_1 + bZ_2 + cZ_3) = 0$$

and Ω, which is clearly invariant under the even part of $\widetilde{\mathrm{D}}(2, 1)$, is $\widetilde{\mathrm{D}}(2, 1)$-invariant.

Since Ω is symmetric and invariant, it corresponds to an invariant symmetric bilinear form on $\widetilde{\mathrm{D}}(2, 1) \otimes K$ which is clearly nonsingular.

It is easy to see the following congruence modulo $\widetilde{\mathrm{D}}(2, 1) \otimes \widetilde{\mathrm{D}}(2, 1)$:

$$\Omega \equiv \frac{1}{2}(af_1 \otimes f_1 + bf_2 \otimes f_2 + cf_3 \otimes f_3)$$

and the cobracket takes values in $\widetilde{\mathrm{D}}(2, 1) \otimes \widetilde{\mathrm{D}}(2, 1)$. $\square$

Theorem 7.8. *Let $\mathbf{Z}[\sigma_2, \sigma_3]$ be the graded subalgebra of $A = \mathbf{Z}[a, b, c]/(a+b+c)$ generated by $\sigma_2 = ab+bc+ca$ of degree 2 and $\sigma_3 = abc$ of degree 3. Then the character χ_{sup} induced by $\widetilde{\mathrm{D}}(2, 1)$ equipped with the Casimir Ω is a graded algebra homomorphism from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[\sigma_2, \sigma_3]$.*

Moreover χ_{sup} satisfies the following:

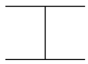
$$\chi_{sup}(t) = 0 \quad and \quad \forall p \geq 0, \quad \chi_{sup}(\omega_p) = \omega\sigma^p$$

with: $\sigma = 4\sigma_2$, $\omega = 8\sigma_3$.

Proof. Since A is a unique factorization domain, we can apply Theorem 6.1 and the character induces by $\widetilde{\mathrm{D}}(2, 1)$ is an algebra homomorphism χ_{sup} from $\Lambda_{\mathbf{Z}}$ to $A = \mathbf{Z}[a, b, c]/(a + b + c)$. There is an action of $\mathfrak{S}_3$ on $\widetilde{\mathrm{D}}(2, 1)$. This action permutes the modules E_i and the coefficients a, b, c. Therefore χ_{sup} takes values in the fixed part of A under the action of $\mathfrak{S}_3$ and χ_{sup} is an algebra homomorphism from $\Lambda_{\mathbf{Z}}$ to $\mathbf{Z}[\sigma_2, \sigma_3]$.

On the other hand, $\widetilde{\mathrm{D}}(2, 1)$ is a graded algebra: elements in $sl(E_i)$ are of degree 0, elements in X are of degree 1 and a, b, c are of degree 2. With this degree the degree of the Lie bracket is 0 and the degree of the cobracket is 2. Hence it is easy to see that each element $u \in \Lambda_{\mathbf{Z}}$ of degree p is sent by χ_{sup} to an element of degree $2p$. Thus, after dividing degrees in A by 2, χ_{sup} becomes a graded character. In particular $\chi_{sup}(t)$ is trivial because $\mathbf{Z}[\sigma_2, \sigma_3]$ has no degree 1 element.

As above denote by Ψ the morphism defined by the diagram



Lemma 7.8.1. *The endomorphism Ψ satisfies the following:*

$$\Psi(\Omega_1) = -4a\Omega_1 + \frac{3}{2}\pi, \qquad \Psi(\Omega_2) = -4b\Omega_2 + \frac{3}{2}\pi, \qquad \Psi(\Omega_3) = -4c\Omega_3 + \frac{3}{2}\pi,$$
$$\Psi(\pi) = -4(a^2\Omega_1 + b^2\Omega_2 + c^2\Omega_3).$$

Proof. We have:

$$\Psi(\Omega_1) = -a\sum_{ij}[e_{1i}, e_{1j}] \otimes [e_{1j}, e_{1i}] + \sum_{ijlk}[e_{1i}, \varepsilon_{1j} \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}] \otimes [\varepsilon'_{1j} \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}, e'_{1i}].$$

The coefficient of $-a$ in this formula is the image of the Casimir of sl_2 under the corresponding homomorphism Ψ_{sl2}. Then it is equal to $2\chi_{sl2}(t)\Omega_1 = 4\Omega_1$, and:

$$\Psi(\Omega_1) = -4a\Omega_1 - \sum_{ijkl}(e_{1i}(\varepsilon_{1j}) \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}) \otimes (e'_{1i}(\varepsilon'_{1j}) \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}).$$

Because of Lemma 7.7.1, we have:

$$\sum_{ij} e_{1i}(\varepsilon_{1j}) \otimes e'_{1i}(\varepsilon'_{1j}) = \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes \varepsilon_{1j}.\varepsilon'_{1i}(\varepsilon'_{1j})$$
$$= \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes (\varepsilon_{1j}\, \varepsilon'_{1i}{\wedge}\varepsilon'_{1j} + \varepsilon'_{1i}\, \varepsilon_{1j}{\wedge}\varepsilon'_{1j})$$
$$= \frac{1}{2}\sum_{j} \varepsilon'_{1j} \otimes \varepsilon_{1j} + \frac{1}{2}\sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i}\sum_{j} \varepsilon_{1j}{\wedge}\varepsilon'_{1j}$$
$$= -\frac{1}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j} - \sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i} = -\frac{3}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j}$$

and that implies the first formula. For computing $\Psi(\Omega_2)$ and $\Psi(\Omega_3)$, just apply a cyclic permutation.

Since Ω is the Casimir and t is zero in this case, we have:

$$0 = \Psi(\Omega) = 4a^2\Omega_1 + 4b^2\Omega_2 + 4c^2\Omega_3 + \Psi(\pi)$$

and that proves the lemma. □

Lemma 7.8.2. *The module $S^2\widetilde{\mathrm{D}}(2, 1) \otimes K$ decomposes into a direct sum $U_0 \oplus U_1 \oplus U_2 \oplus U_3$. The module U_0 is isomorphic to K and generated by the Casimir. The homomorphism Ψ respects this decomposition. It acts on U_0, U_1, U_2, U_3 by multiplication by $0, 2a, 2b, 2c$ respectively.*

Proof. Set: $L = \widetilde{\mathrm{D}}(2,1) \otimes K$. Let V_0 be the K-submodule of S^2L generated by $\Omega_1, \Omega_2, \Omega_3, \pi$. The morphism Ψ induces an endomorphism of V_0. The matrix of this endomorphism in the basis $(2\Omega_1, 2\Omega_2, 2\Omega_3, \pi)$ is:

$$\begin{pmatrix} -4a & 0 & 0 & -2a^2 \\ 0 & -4b & 0 & -2b^2 \\ 0 & 0 & -4c & -2c^2 \\ 3 & 3 & 3 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $0, 2a, 2b, 2c$ and corresponding eigenvectors are:

$$\Omega = -a\Omega_1 - b\Omega_2 - c\Omega_3 + \pi$$
$$2a(b-c)\Omega_1 + 6b^2\Omega_2 - 6c^2\Omega_3 - 3(b-c)\pi$$
$$2b(c-a)\Omega_2 + 6c^2\Omega_3 - 6a^2\Omega_1 - 3(c-a)\pi$$
$$2c(a-b)\Omega_3 + 6a^2\Omega_1 - 6b^2\Omega_2 - 3(a-b)\pi.$$

Let L_0 be the even part of L. Let F_p be the simple sl_2-module of dimension $p+1$. This module is the symmetric power S^pF_1 and $F_2 = sl_2$. Denote by $[p,q,r]$ the isomorphism class of the L_0-module $F_p \otimes F_q \otimes F_r$. These elements form a basis of the Grothendieck algebra $\mathrm{Rep}(L_0)$ of representations of L_0. In this algebra we have:

$$[L_0] = [2,0,0] + [0,2,0] + [0,0,2] \qquad [X] = [1,1,1]$$
$$[S^2L_0] = 3[0,0,0] + [4,0,0] + [0,4,0] + [0,0,4] + [2,2,0] + [2,0,2] + [0,2,2]$$
$$[\Lambda^2X] = [0,0,0] + [2,2,0] + [2,0,2] + [0,2,2]$$
$$[L_0 \otimes X] = 3[1,1,1] + [3,1,1] + [1,3,1] + [1,1,3].$$

The module V_0 is the submodule $3[0,0,0] + [0,0,0]$ of S^2L. Set $V'_0 = V_0$ and define by induction submodules V'_p to be the image of $X \otimes V'_{p-1}$ under the action map. Then set: $V_p = V'_0 + \cdots + V'_p$. For every $p \geq 0$, V_p is a L_0-module. It is not difficult to prove the following:

$$[V_0] = 4[0,0,0] \qquad [V_1] = 4[0,0,0] + 3[1,1,1]$$
$$[V_2] = 4[0,0,0] + 3[1,1,1] + [2,2,0] + [2,0,2] + [0,2,2] \implies \Lambda^2X \subset V_2$$
$$[V_3] = 4[0,0,0] + 3[1,1,1] + [2,2,0] + [2,0,2] + [0,2,2] + [3,1,1] + [1,3,1] + [1,1,3]$$
$$\implies \Lambda^2X \oplus L_0 \otimes X \subset V_3.$$

Then there is a unique L_0-submodule W of $S^2L_0 \subset S^2L$ such that $V_3 \oplus W = S^2L$. If V is the L-submodule of S^2L generated by V_0, the module S^2L/V is a quotient of W and then has no odd degree component. Therefore this module is trivial and S^2L is generated by V_0 as a L-module, and that implies that S^2L is the direct sum of L-modules generated by the eigenvectors above and the lemma is proven. □

Now we are able to apply Theorem 6.3 and we get the desired result. □

Remark. There exist an extra Lie superalgebra equipped with a Casimir element: the Hamiltonian algebra $H(n)$ for $n > 4$ and n even and that is a complete list of simple quadratic Lie superalgebras [12]. For $n > 4$ the Hamiltonian algebra $L = H(n)$ has the following property: it has a $\mathbf{Z}$-graduation compatible with the Lie bracket, and the Casimir has a nonzero degree. Therefore for any element $u \in \Lambda$ of positive degree, the induced element $\chi_L(u)$ has a nonzero degree. But it is an element of the coefficient field. Then χ_L vanishes on positive degree elements and χ_L is the augmentation character.

8. Properties of the characters

In the last section, we constructed eight characters χ_i, $i = 1\dots 8$ corresponding to families gl, osp, E_6, E_7, E_8, F_4, G_2 and $\widetilde{\mathrm{D}}(2,1)$. These characters are graded algebra homomorphisms from Λ to A_i, where $A_1 = \mathbf{Q}[t,u]$, $A_2 = \mathbf{Q}[t,v]$, $A_3 = A_4 = A_5 = A_6 = A_7 = \mathbf{Q}[t]$, $A_8 = \mathbf{Q}[\sigma_2, \sigma_3]$.

Consider the subalgebra $R_0 = \mathbf{Q}[t] \oplus \omega\mathbf{Q}[t,\sigma,\omega]$ of $R = \mathbf{Q}[t,\sigma,\omega]$. This algebra is sent to Λ by a morphism φ defined by:

$$\varphi(t) = t, \qquad \forall p \geq 0, \qquad \varphi(\sigma^p\omega) = \omega_p.$$

For each $i = 1, \dots, 8$ there is a unique character χ'_i from R to A_i which restricts on R_0 to $\chi_i \circ \varphi$. These morphisms are defined by:

$$\chi'_1(t) = t \qquad \chi'_1(\sigma) = 2(t^2 - 2u) \qquad \chi'_1(\omega) = 2t(t^2 - 4u)$$
$$\chi'_2(t) = t \qquad \chi'_2(\sigma) = 3(t - 2v)(t + 3v) \qquad \chi'_2(\omega) = 2(t - v)(t - 2v)(t + 4v)$$
$$\chi'_3(t) = t \qquad \chi'_3(\sigma) = \frac{77}{36}t^2 \qquad \chi'_3(\omega) = \frac{25}{12}t^3$$

$$\chi'_4(t) = t \qquad \chi'_4(\sigma) = \frac{176}{81}t^2 \qquad \chi'_4(\omega) = \frac{520}{243}t^3$$
$$\chi'_5(t) = t \qquad \chi'_5(\sigma) = \frac{494}{225}t^2 \qquad \chi'_5(\omega) = \frac{98}{45}t^3$$
$$\chi'_6(t) = t \qquad \chi'_6(\sigma) = \frac{170}{81}t^2 \qquad \chi'_6(\omega) = \frac{480}{243}t^3$$
$$\chi'_7(t) = t \qquad \chi'_7(\sigma) = \frac{65}{36}t^2 \qquad \chi'_7(\omega) = \frac{55}{36}t^3$$
$$\chi'_8(t) = 0 \qquad \chi'_8(\sigma) = 4\sigma_2 \qquad \chi'_8(\omega) = 8\sigma_3.$$

The kernels of these characters are:

$$I_1 = \operatorname{Ker}\chi'_1 = (P_{gl})$$
$$I_2 = \operatorname{Ker}\chi'_2 = (P_{osp})$$
$$I_3 = \operatorname{Ker}\chi'_3 = (P_{exc}, 77t^2 - 36\sigma)$$
$$I_4 = \operatorname{Ker}\chi'_4 = (P_{exc}, 176t^2 - 81\sigma)$$
$$I_5 = \operatorname{Ker}\chi'_5 = (P_{exc}, 494t^2 - 225\sigma)$$
$$I_6 = \operatorname{Ker}\chi'_6 = (P_{exc}, 170t^2 - 81\sigma)$$
$$I_7 = \operatorname{Ker}\chi'_7 = (P_{exc}, 65t^2 - 36\sigma)$$
$$I_8 = \operatorname{Ker}\chi'_8 = (t)$$

with:

$$P_{gl} = \omega - 2t\sigma + 2t^3$$
$$P_{osp} = 27\omega^2 - 72t\sigma\omega + 40t^3\omega + 4\sigma^3 + 29t^2\sigma^2 - 24t^4\sigma$$
$$P_{exc} = 27\omega - 45t\sigma + 40t^3.$$

Using the inclusion $\mathbf{Q}[t, \sigma, \omega] \subset \mathbf{Q}[\alpha, \beta, \gamma]$ we check the following:

$$P_{gl} = (\alpha - t)(\beta - t)(\gamma - t) = -(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$$
$$P_{osp} = (\alpha + 2\beta)(2\alpha + \beta)(\beta + 2\gamma)(2\beta + \gamma)(\gamma + 2\alpha)(2\gamma + \alpha)$$
$$P_{exc} = (3\alpha - 2t)(3\beta - 2t)(3\gamma - 2t).$$

Since characters χ'_i are surjective each character χ_i may be consider as a graded algebra homomorphism from Λ to a quotient of R. These characters are related. The complete relations between them are given by the following result of Patureau-Mirand:

Theorem 8.1 ([18]). *Let I be the following ideal in R:*

$$I = t\omega P_{gl}P_{osp}(P_{exc}, (77t^2 - 36\sigma)(176t^2 - 81\sigma)(494t^2 - 225\sigma)(170t^2 - 81\sigma)(65t^2 - 36\sigma)).$$

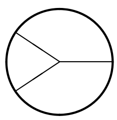
Then there is a unique graded algebra homomorphism χ from Λ to R_0/I such that:

$$\chi_{sl2} \equiv \chi \bmod \omega R$$
$$\forall i = 1 \quad \dots 8, \qquad \chi_i \equiv \chi \bmod I_i.$$

Remark. It was conjectured in [1] that every element in $\mathcal{A}$ is detected by invariants coming from Lie algebras in series A, B, C, D. This conjecture is false. There is a weaker conjecture saying that invariants coming from simple Lie algebras detect every element in $\mathcal{A}$. That is also false because of the Lie superalgebra $\widetilde{\mathrm{D}}(2, 1)$. Actually we have the following result:

Theorem 8.2. *There exists a primitive element in $\mathcal{A}$ of degree 17 which is rationally nontrivial and killed by every weight function obtained by a semisimple Lie (super)algebra and a finite-dimensional representation.*

Proof. Let u be the following primitive element of $\mathcal{A}$ of degree 2:



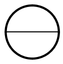
The map $\lambda \mapsto \lambda u$ is a rational injection from Λ to the module $\mathcal{P}$ of primitives of $\mathcal{A}$ (see Corollary 4.7). Let U be the image of

$P = \omega P_{gl} P_{osp} P_{exc}$ under the morphism $\varphi : R_0 \longrightarrow \Lambda$. This element is detected by χ_8 and is rationally nontrivial. Then Uu is an element rationally nontrivial in $\mathcal{A}$ of degree 17.

Let L be a simple Lie superalgebra equipped with a Casimir element. If L is of type $i \neq 8$ we have:

$$\Phi_L(Uu) = \chi_i'(P)\Phi_L(u) = 0.$$

If L is of type 8 (i.e. $L = \widetilde{\mathrm{D}}(2,1)$), we have:

$\Phi_L(Uu) = \chi_8'(P)\Phi_L(u) = \chi_8'(P)\chi_8(t)\Phi_L($  $).$

But $\chi_8(t) = 0$. Therefore Uu is killed by Φ_L.

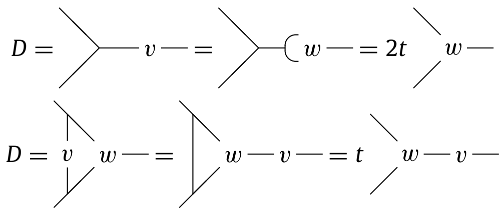
If $L = \oplus L_i$ is semisimple, $\Phi_L(Uu) = \sum \Phi_{L_i}(Uu) = 0$ because Uu is primitive. □

Theorem 8.3. *Let u be an element in Λ killed by $\chi_1, \chi_2, \ldots, \chi_8$. Let L be a quadratic Lie superalgebra over a field of characteristic 0. Then u is killed by Φ_L.*

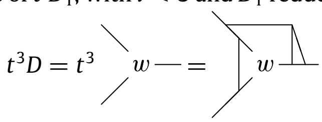
Proof. Let D be a connected diagram in $\mathcal{D}(\emptyset, [3])$ representing some element u' in $F(3)$. Let D_0 be the union of closed edges meeting ∂K and D_1 be the complement of D_0 in K. We will say that D is reduced if D_1 is connected.

Lemma 8.3.1. *Every connected diagram in $\mathcal{D}(\emptyset, [3])$ of degree > 2 is equivalent in $F(3)$ to a multiple of a reduced diagram.*

Proof. Let d be the degree of a connected diagram D. If d is positive and D is not reduced, we have the following possibilities in $F(3)$ (up to some cyclic permutation in $\mathfrak{S}_3$):



Therefore D is equivalent in $F(3)$ to a multiple of $t^i D_1$, with $i < 3$ and D_1 reduced or $i = 3$. But it is easy to see the following:



Since a reduced diagram multiply by t is represented by a reduced diagram, the result follows. □

Since χ_{gl} detects every element in Λ in degree < 6, we may suppose that u is an element in Λ of degree $d \geq 6$. Consider the category of diagrams Δ. Any element in $F(m)$ may be seen as a morphism in Δ from $\emptyset$ to $[m]$. Let β be the bracket from [2] to [1] (β is represented by a tree). Because of the lemma, there is an element $v \in F(6)$ such that:

$$u = \beta^{\otimes 3} \circ v.$$

Moreover the degree of v is $d - 3 > 2$.

Consider now a quadratic Lie superalgebra L over a field of characteristic zero and a central extension E of L. Denote by K the kernel of $E \longrightarrow L$. The Lie bracket $E \otimes E \longrightarrow E$ is trivial on $K \otimes E + E \otimes K$ and induces an extended bracket $\psi : L \otimes L \longrightarrow E$. So we can set:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) \in E^{\otimes 3}.$$

Lemma 8.3.2. *Let I be an ideal in L and $I^\perp$ be its orthogonal. Let E_1 and E_2 be the pullback in E of I and $I^\perp$. Suppose that the inner form is nonsingular on I. Then we have:*

$$\Phi_{E,L}(u) = \Phi_{E_1,I}(u) + \Phi_{E_2,I^\perp}(u).$$

Proof. The modules I and $I^\perp$ are Lie superalgebras. Since the form is nonsingular on I, L is the direct sum $I \oplus I^\perp$. It is easy to see that I and $I^\perp$ are quadratic Lie superalgebras and $E_1 \longrightarrow I$ and $E_2 \longrightarrow I^\perp$ are central extensions. Then we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = \psi^{\otimes 3}(\Phi_I(v) + \Phi_{I^\perp}(v)) = \Phi_{E_1,I}(u) + \Phi_{E_2,I^\perp}(u). \quad \square$$

Lemma 8.3.3. *Let I be an isotropic ideal of L and $I^\perp$ be its orthogonal. Let J be the quotient $I^\perp/I$. Suppose that the form on J induced by the inner form on L is nonsingular on the center of J. Let E_1 be the pullback in E of the module $[I^\perp, I^\perp] \subset L$. Then E_1 is a central extension of $J_1 = [J, J]$ and we have:*

$$\Phi_{E,L}(u) = \Phi_{E_1,J_1}(u).$$

Proof. Since I is a L-module, $I^\perp$ and $J = I^\perp/I$ are L-modules too. Moreover for any $(x, y, z) \in I \times I^\perp \times L$ we have:

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle = 0$$

and $[x, y]$ is orthogonal to every $z \in L$. Then $[x, y] = 0$ for every $x \in I$ and $y \in I^\perp$ and the bracket is trivial on $I \otimes I^\perp$. Therefore the Lie bracket and the inner form induce a quadratic Lie superalgebra structure on $J = I^\perp/I$.

The central extension $I^\perp \longrightarrow J$ is determined by a 2-cocycle $\varphi : \Lambda^2 J \longrightarrow I$. The cohomology class of φ is determined by a morphism $\mathrm{H}_2(J) \longrightarrow I$ and it is possible to modify φ by a coboundary in such a way that φ and $\mathrm{H}_2(J) \longrightarrow I$ have the same image. Then $I^\perp$ can be identify to $I \oplus J$ and the Lie bracket $[\ ,\ ]_1$ on $I \oplus J$ is given by:

$$\forall \alpha, \beta \in I, \forall x, y \in J, \quad [\alpha + x, \beta + y]_1 = [x, y] + \varphi(x \otimes y)$$

where φ is a cocycle satisfying: $\varphi(\Lambda^2(J)) = \varphi(\mathrm{Ker}(\Lambda^2 J \to J))$. The central extension induces an extended bracket ψ' from $\Lambda^2(J)$ to $I^\perp$.

Since the form is nonsingular on J, it is nonsingular on its orthogonal $J^\perp$. Then there exists a module $I^* \subset L$ such that the form is trivial on I^* and $J^\perp$ is the module $I \oplus I^*$. Therefore the Casimir element Ω decomposes into a sum: $\Omega = \Omega_0 + \Omega_+ + \Omega_-$, where Ω_0, Ω_+ and Ω_- are in $J \otimes J$, $I \otimes I^*$ and $I^* \otimes I$ respectively.

Suppose $v \in F(6)$ is represented by a connected diagram D such that the edges of D meeting ∂D are disjoint. Therefore there exists a diagram D' representing an element w in $\mathcal{A}(\emptyset, [12])$ such that:

$$v = \beta^{\otimes 6} \circ w.$$

Actually every element in $F(6)$ of positive degree is a linear combination of such diagrams.

Set $\partial D = \{v_i\}$, $i = 1 \ldots 6$ and denote by e_i the oriented edge in D starting from v_i. Let U be the set of oriented subgraphs of D and Γ be an oriented graph in U. For each oriented edge $a \in D$ define the module $V_\Gamma(a)$ by:

$$V_\Gamma(a) = \begin{cases} I^* & \text{if } a \in \Gamma \\ I & \text{if } -a \in \Gamma \\ J & \text{otherwise.} \end{cases}$$

Let e be an edge in D and a and $-a$ be the corresponding oriented edges. Set $\Omega_\Gamma(e)$ be the component of the Casimir element Ω in $V_\Gamma(a) \otimes V_\Gamma(-a)$ and denote by $\Omega(\Gamma)$ the tensor product:

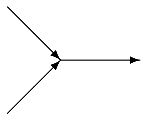
$$\Omega(\Gamma) = \bigotimes_e \Omega_\Gamma(e).$$

For each 3-valent vertex x in D the alternating form $\langle\ ,\ ,\ \rangle$ induces a linear form on $V_\Gamma(a) \otimes V_\Gamma(b) \otimes V_\Gamma(c)$ where a, b, c are the three oriented edges in D starting from x. By applying all these forms to $\Omega(\Gamma)$ we get an element $\Phi(\Gamma)$ in $\otimes_i V_\Gamma(e_i)$. It is not difficult to see that $\Phi_L(v)$ is the sum of all $\Phi(\Gamma)$.

Let x be a 3-valent vertex in D and a, b, c be the oriented edges in D ending at x. Since the alternating form $\langle x, y, z\rangle$ vanishes for $x \in I$ and $y \in I \oplus J$, $\Omega(\Gamma)$ is zero if a is in Γ and $-b$ (or $-c$) is not in Γ.

Denote by U_+ the set of all Γ in U such that:

for every 3-valent vertex v in D, if one oriented edge starting from v is in Γ the two other edges ending at v are in Γ too.



Then we have:

$$\Phi_L(v) = \sum_{\Gamma \in U_+} \Phi(\Gamma).$$

Let Γ be an oriented graph in U_+. Suppose Γ contains some oriented edge e disjoint from ∂D. Since $D \setminus \{e_i\}$ is a connected 3-valent graph, there is a long oriented path $(f_1, f_2, \ldots, f_p = e)$ in D such that each oriented edge f_j is in Γ. Therefore Γ contains an oriented cycle C. Since the degree of v is at least 2, there exist an edge e' outside of $\{e_i\}$ and meeting C in some vertex v. Then there is a long oriented path $(g_1, g_2, \ldots, g_q)$ such that g_q is the edge e' ending at v. But this path is necessary included in D because Γ is in U_+ and that is impossible. Hence Γ has to be included in $\{e_i\}$ with the right orientation.

Then we have:

$$\Phi_L(v) = \psi'^{\otimes 6}(\Phi_J(w)).$$

Let J_2 be the center of J. Since the form is nonsingular, J is the direct sum: $J = J_1 \oplus J_2$. The center of J_1 is trivial and then: $J_1 = [J_1, J_1]$. Since J_2 is abelian, we have:

$$\Phi_L(v) = \psi'^{\otimes 6}(\Phi_J(w)) = \psi'^{\otimes 6}(\Phi_{J_1}(w) + \Phi_{J_2}(w)) = \psi'^{\otimes 6}(\Phi_{J_1}(w)).$$

Let I_1 be the image of φ. Then the module $[I^\perp, I^\perp]$ is $[I \oplus J, I \oplus J]_1 = I_1 \oplus J_1$. Denote by φ_1 a 2-cocycle on $\Lambda^2 L$ which determines the extension $E \longrightarrow L$. Let $\alpha \in I_1$ and x and y in J. Since φ_1 is a cocycle, we have:

$$\varphi_1(\alpha \otimes [x, y]_1) = -\varphi_1(x \otimes [y, \alpha]_1) - \varphi_1(y \otimes [\alpha, x]_1) = 0$$
$$\Longrightarrow \varphi_1(\alpha, [x, y] + \varphi(x \otimes y)) = 0.$$

Then if w lies in the kernel of $\Lambda^2 J \to J$, we have: $\varphi_1(\alpha, \varphi(w)) = 0$ and φ_1 is trivial on $I_1 \otimes I_1$ and therefore on $I_1 \otimes J_1$. Hence the cocycle on $[I^\perp, I^\perp]$ comes from a cocycle on J_1 and the extension $E_1 \longrightarrow J_1$ is central. This extension induces an extended bracket $\psi'' : \Lambda^2 J_1 \longrightarrow E_1$ and we have for every x_1, x_2, x_3, x_4 in J_1:

$$\psi(\psi'^{\otimes 2}(x_1 \otimes x_2 \otimes x_3 \otimes x_4)) = \psi(\psi'(x_1 \otimes x_2) \otimes \psi'(x_3 \otimes x_4)) = \psi''([x_1, x_2] \otimes [x_3, x_4])$$
$$\Longrightarrow \psi \circ \psi'^{\otimes 2} = \psi'' \circ \beta^{\otimes 2}$$

where β is the Lie bracket on L_1. Therefore we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = \psi^{\otimes 3}(\psi'^{\otimes 6}(\Phi_{J_1}(w)))$$
$$= \psi''^{\otimes 3}(\beta^{\otimes 3}(\Phi_{J_1}(w))) = \psi''^{\otimes 3}(\Phi_{J_1}(v)) = \Phi_{E_1,J_1}(u). \quad \square$$

Now we are able to prove that $\Phi_{E,L}(u)$ is zero by induction on $\dim(E) + \dim(L)$.

Let E be a central extension of a quadratic Lie superalgebra L. Suppose there is some nontrivial ideal in L contained in its orthogonal. Let I be such a maximal ideal. Set: $J = I^\perp/I$. Since I is maximal, J does not contain any nontrivial isotropic ideal and the inner form on J is nonsingular on the center of J. Hence $\Phi_{E,L}(u)$ is trivial by induction, because of Lemma 8.3.3.

Suppose L has some nontrivial simple submodule I. The inner form is now nonsingular on I and $\Phi_{E,L}(u)$ is trivial by induction, because of Lemma 8.3.2.

So we have to suppose that L is simple. If L is isomorphic to some $sl(E)$, $osp(E)$, $E6$, $E7$, $E8$, $F4$, $G2$, $G(3)$, $F(4)$ or $D(2, 1, \alpha)$, the cohomology of L is isomorphic to the cohomology of some semisimple Lie algebra [10] and $\mathrm{H}^2(L)$ is trivial. Therefore the extension $E \longrightarrow L$ is trivial and has a section s. So we have:

$$\Phi_{E,L}(u) = \psi^{\otimes 3}(\Phi_L(v)) = s^{\otimes 3} \circ \beta^{\otimes 3}(\Phi_L(v)) = s^{\otimes 3}(\Phi_L(u))$$

and this element is trivial because u is killed by each character χ_i.

If L is isomorphic to some $psl(E)$, $\mathrm{H}^2(L)$ is a 1-dimensional module generated by the central extension $sl(E) \longrightarrow psl(E)$. Then there is a morphism $s : sl(E) \longrightarrow psl(E) = L$ and then this extension factorizes through E. So we have:

$$\Phi_{E,L}(u) = s^{\otimes 3}(\Phi_{sl(E),L}(u))$$

and $\Phi_{E,L}(u)$ is detected by $\Phi_{sl(E),L}(u)$ and then by $\Phi_{gl(E)}(u)$. Therefore $\Phi_{E,L}(u)$ is trivial because $\Phi_{gl(E)}(u)$ is detected by $\chi_1 = \chi_{gl}$.

In the last possibility L is isomorphic to an Hamiltonian Lie superalgebra $H(n)$ with $n = 2p > 4$. Consider the Hamiltonian Lie superalgebra $E_0 = \widehat{H}(n)$ and its commutator $E_1 = [\widehat{H}(n), \widehat{H}(n)]$ (see the Appendix). Since $\mathrm{H}^2(H(n))$ is 1-dimensional and generated by the central extension $E_1 \longrightarrow H(n)$, there is a morphism $s : E_1 \longrightarrow E$ and this extension factorizes through E. So we have:

$$\Phi_{E,L}(u) = s^{\otimes 3}(\Phi_{E_1,L}(u))$$

and $\Phi_{E,L}(u)$ is detected by $\Phi_{E_1,L}(u)$ and then by $\Phi_{E_0}(u)$. But $E_0 = \widehat{H}(n)$ is $\mathbf{Z}$-graded and the degree of its cobracket is $n - 4$. Then $\Phi_{E_0}(u)$ is an element in $\Lambda^3 E_0$ of degree $d(n-4)$. On the other hand E_0 is concentrated in degrees $-2, -1, \ldots, n-2$ and $\Lambda^3 E_0$ is concentrated in degrees $-5, -4, \ldots, 3n-7$. If $\Phi_{E_0}(u)$ is nonzero we have:

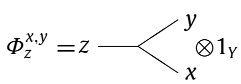
$$d(n-4) \le 3n - 7 \Longrightarrow (d-3)(n-4) \le 5 \Longrightarrow 2(d-3) \le 5 \Longrightarrow d \le 5.$$

But that is not true and $\Phi_{E,L}(u)$ is trivial. $\square$

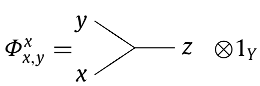
Theorem 8.4. *Let J be the ideal of R generated by $t\omega P_{gl} P_{osp} P_{exc}$. Then J is killed by the morphism $\varphi : R_0 \longrightarrow \Lambda$.*

Proof. Let Δ' be the monoidal subcategory of Δ generated by diagrams where each component meets source and target. Let X be a finite set. If x and y are distinct points in X we may define three morphisms in the category Δ' in the following way:

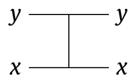
Denote by Y the complement: $Y = X \setminus \{x, y\}$. Take a point z (outside of Y) and set: $Z = Y \cup \{z\}$. So we define a morphism $\Phi_z^{x,y}$ from X to Z by:



We have also a morphism $\Phi_{x,y}^z$ from Z to X defined by:



and a morphism $\Psi_{x,y}$ from X to X defined by:

$\Psi_{x,y} =$  $\otimes 1_Y$

The set of all these morphisms will be denoted by $\mathcal{M}$.

Let f be one of these morphisms. The set $\{x, y, z\}$ in the first two cases or the set $\{x, y\}$ in the last case will be called the support of f. Using this terminology we have the following relations:
R1: if f and g are two composable morphisms in $\mathcal{M}$ with disjoint support they commute.
R2: $\Psi_{x,y} = \Phi_y^{z,y} \circ \Phi_{x,z}^x$
R3: $\Psi_{x,y} - \Psi_{x,y} \circ \tau_{x,y} = \Phi_{x,y}^z \circ \Phi_z^{x,y}$, where $\tau_{x,y}$ is the transposition $x \leftrightarrow y$.

Let X be a finite set and x be an element in X. Denote by Y the complement $Y = X \setminus \{x\}$. We have the following morphisms:

$$\Phi_x = \sum_{y\in Y} \Phi_{x,y}^y, \qquad \Phi^x = \sum_{y\in Y} \Phi_y^{x,y}, \qquad \Psi_x = \sum_{y\in Y} \Psi_{x,y}.$$

They are morphisms from X to Y, Y to X and X to X respectively.

The collection of modules $F'(X) = \mathcal{A}^s(\emptyset, X)$ define a Δ'-module F. Because of Lemma 3.3 it is easy to see that Φ_x and Φ^x act trivially on F and Ψ_x acts on F by multiplication by $2t$. So we may define a new category $\widetilde{\Delta}$: the objects in this category are nonempty finite sets and the morphisms are $\mathbf{Q}[t]$-modules defined by generators and relations where the generators are the bijections in finite sets and the elements in $\mathcal{M}$ and the relations are the following:

– relations R1, R2, R3

– $\Phi_x = 0$, $\Phi^x = 0$ and $\Psi_x = 2t$ for each point x in some finite set.

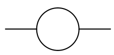
This category contains the category $\mathfrak{S}$ of finite sets and bijections and F is a $\widetilde{\Delta}$-module.

Let $n > 1$ be an integer. Denote by Δ_n the category of finite sets with cardinal in $\{1, 2, \ldots, n\}$ and morphisms defined by generators and relations:

– generators: bijections and elements in $\mathcal{M}$ involving only sets of cardinal $\leq n$

– relations: relations in $\widetilde{\Delta}$ involving only sets of cardinal $\leq n$.

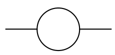
By restriction F induces a Δ_n-module F_n. For example $F_2(X)$ is trivial if $\#X \neq 2$ and is the free module generated by:

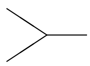


otherwise.

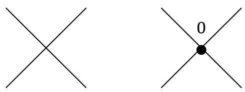
Define the Δ_4-module G_4 by:

– $G_4(X) = 0$ if $\#X = 1$

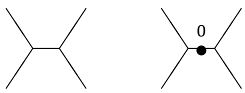
– if $\#X = 2$, $G_4(X)$ is the free R_0-module generated by 

– if $\#X = 3$, $G_4(X)$ is the free R_0-module generated by 

– if $\#X = 4$, $G_4(X)$ is a direct sum $R_0 \otimes U_1 \oplus R \otimes U_2 \oplus R_0 \otimes V_1 \oplus R \otimes V_2$, where V_1 and V_2 are 1-dimensional modules generated by the following diagrams:



and U_1 and U_2 are 2-dimensional simple $\mathfrak{S}_4$-modules generated by the following diagrams:



The action of the category $\widetilde{\Delta}_4$ on this module is defined by Proposition 5.5.

For each $n > 4$ define the module G_n by scalar extension:

$$G_n = \Delta_n \bigotimes_{\Delta_{n-1}} G_{n-1}.$$

These modules can be determined by computer for small values of n. For every Young diagram α of size n denote by $V(\alpha)$ a simple $\mathfrak{S}_n$-module corresponding to α. If X is a finite set of cardinal p, $G_n(X)$ is a $\mathfrak{S}_p$-module and we get the following:

– if $p \leq 4$, $G_4(X) \simeq G_5(X) \simeq G_6(X)$

– if $p = 5$, $G_5(X)$ and $G_6(X)$ are isomorphic to

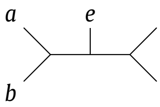
$$(R_0 \oplus R \oplus R) \otimes V(3, 1, 1) \oplus R \otimes V(2, 1, 1, 1)$$

– if $p = 6$, $G_6(X)$ is isomorphic to

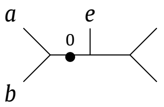
$$(R_0 \oplus R^5) \otimes (V(4,2) \oplus V(2,2,2)) \oplus (R_0 \oplus R^3) \otimes V(6)$$
$$\oplus R^2 \otimes (V(3,2,1) \oplus V(2,1,1,1,1)) \oplus R \otimes (V(3,1,1,1) \oplus V(5,1)).$$

For a complete description of $G_6(X)$ in the case $p = 5$ we may proceed as follows:

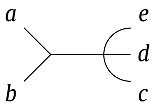
Let $E(X)$ be the **Q**-vector space generated by the elements of X with the single relation: $\sum_{x \in X} x = 0$. Then $\Lambda^2 E(X)$ and $\Lambda^3 E(X)$ are simple modules corresponding to Young diagrams $(3,1,1)$ and $(2,1,1,1)$ and we can set: $V(3,1,1) = \Lambda^2 E(X)$ and $V(2,1,1,1) = \Lambda^3 E(X)$. So with the identification $G_6(X) = (R_0 \oplus R \oplus R) \otimes V(3,1,1) \oplus R \otimes V(2,1,1,1)$, we have the following:



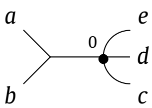
$= A \otimes (a - b) \wedge (d - c),$



$= C \otimes (a - b) \wedge (d - c) + D \otimes (a - b) \wedge d \wedge c,$



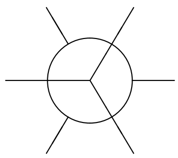
$= (\frac{10}{3} tA + B) \otimes a \wedge b,$



$= (\frac{10}{3} tC + \sigma B) \otimes a \wedge b,$

where A generates a free R_0-module and B, C, D generate free R-modules.

Let X be a set of cardinal 6. Consider the element U in $F(X)$ represented by the following diagram:



This element is not in G_6 but it corresponds to en element U_0 in G_7. With the following idempotent in $\mathbf{Q}[\mathfrak{S}_X]$:

$$\pi = \frac{1}{6!} \sum_{\sigma \in \mathfrak{S}_X} \varepsilon(\sigma)\sigma$$

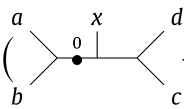
we can set: $V = \pi U$ and $V_0 = \pi U_0$.

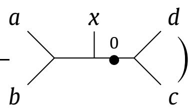
Let x and y be two distinct elements in X. Set: $Y = X \setminus \{y\}$ and $Z = X \setminus \{x, y\}$. We have:

$$tV_0 = \frac{1}{2} \Psi_x V_0 = \frac{1}{4} \sum_{z \neq x} \Phi^x_{x,z} \circ \Phi^{x,z}_x V_0 = \frac{\pi}{4} \sum_{z \neq x} \Phi^x_{x,z} \circ \Phi^{x,z}_x V_0 = \frac{5\pi}{4} \Phi^x_{x,y} \circ \Phi^{x,y}_x V_0.$$

It is not difficult to see that $\Phi^{x,y}_x V_0$ is an element in $G_6(Y)$ completely antisymmetric in Z.

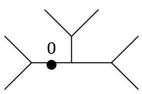
Let a, b, c, d be the elements in Z. It is easy to see that every element in $V(3,1,1)$ completely antisymmetric in a, b, c, d is trivial and any element in $V(2,1,1,1) = \Lambda^3 E(Y)$ completely antisymmetric in a, b, c, d is a multiple of $a \wedge b \wedge c - a \wedge b \wedge d + a \wedge c \wedge d - b \wedge c \wedge d$. Therefore there is an element P in R such that:

$\Phi^{x,y}_x V_0 = P\Big($ 

$-$ 

$\Big).$

But for a diagram like this:



there is a double transposition in $\mathfrak{S}_6$ which acts on it by multiplication by -1 and its antisymmetrization is trivial. Therefore V_0 and V are killed by t.

On the other hand there is a pairing on each $F'(X)$ with values in Λ:

if u and u' are two elements in $F'(X)$ represented by diagrams D and D', we can glue D and D' along X and we get a connected diagram D_1. The class of D_1 in $F(0)$ is the multiple of the Theta diagram by some element $\lambda \in \Lambda$. So we set: $\langle u, u'\rangle = \lambda$.

Consider the element $P = \langle U, V\rangle$ in Λ. This element is of degree 15. Since V is killed by t, we have in Λ the relation: $tP = 0$.

On the other hand we can check by computer that the morphism $G_6(X) \longrightarrow G_7(X)$ is surjective for $\#X < 6$. So P lies in a quotient of R_0 and P can be seen as an element in R_0.

Since $tP = 0$, P is killed by χ_{sl2}, χ_{gl} and χ_{osp}. So we have:

$$P = \omega P_{gl} P_{osp} Q$$

for some $Q \in R$ of degree 3. But Q is also killed by the exceptional characters χ_i and Q is a multiple of P_{exc}. At the end we get:

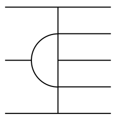
$$P = k\,\omega P_{gl} P_{osp} P_{exc}$$

for some rational k. A direct computation (by computer) gives the following result:

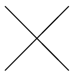
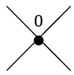
$$P = 2^{-10}\omega P_{gl} P_{osp} P_{exc} \implies t\omega P_{gl} P_{osp} P_{exc} = 0 \in \Lambda.$$

One can also determine P by using the Lie superalgebra $\widetilde{D}(2, 1)$.

Consider the morphism A in Δ_6 defined by the diagram:



The morphism $B = 1 \otimes A$ may be consider as a morphism from X to a set Z of cardinal 4. Let π' be the sum of all elements in $\mathfrak{S}_Z$ divided by 4!. Since B lies in Δ_7 the element $\pi' \circ B.V_0$ belongs to $\Delta_7(Z)$ and can be seen as an element W in $G_6(Z) = G_4(Z)$. Since $\mathfrak{S}_Z$ acts trivially on W, there are two elements $Q \in R_0$ and $Q' \in R$ such that $W = QH + Q'H'$ with:

$H =$  $H' =$ 

Degrees of Q and Q' are 12 and 10 respectively. Since tV_0 is trivial W is killed by t in $F(Z)$ and W is killed by Φ_{sl2}, Φ_{sl_n} and Φ_{o_n}.

The functor Φ_{sl2} kills H' but not H. Then Q is killed by χ_{sl2}.

For n big enough the vectors $\Phi_{sl_n}(H)$ and $\Phi_{sl_n}(H')$ are linearly independent. Then Q and Q' are killed by χ_{gl}.

The same holds for Φ_{o_n} and Q and Q' are killed by χ_{osp}.

Thus there exist c and c' in $\mathbf{Q}$ with: $Q = c\omega P_{gl} P_{osp}$ and $Q' = c' t P_{gl} P_{osp}$.

Let L be an exceptional Lie algebra. Then we have:

$$\Phi_L(H') = \frac{3\omega}{5t}\Phi_L(H) \implies \chi_L(5tQ) + \chi_L(3Q') = 0 \implies c' = -5/3c.$$

On the other hand we have:

$$\begin{aligned} P &= \langle H, W\rangle = c\omega P_{gl} P_{osp}\langle H, H\rangle - 5/3ctP_{gl} P_{osp}\langle H, H'\rangle \\ &= cP_{gl} P_{osp}(\omega\langle H, H\rangle - 5/3t\langle H, H'\rangle) \end{aligned}$$

and for every $p \geq 0$:

$$0 = \langle \sigma^p H', tW\rangle = tcP_{gl} P_{osp}(\omega\sigma^p\langle H', H\rangle - 5/3t\sigma^p\langle H', H'\rangle).$$

Since P is nonzero c is nonzero too. So we have:

$$t\sigma^p P_{gl} P_{osp}(\omega\langle H', H\rangle - 5/3t\langle H', H'\rangle) = 0.$$

A direct computation gives:

$$\langle H', H\rangle = -\frac{3}{2}\sigma\omega + \frac{10}{3}t^2\omega, \qquad \langle H', H'\rangle = -\frac{3}{2}\sigma^2\omega + \frac{4}{3}t^2\sigma\omega + 2t\omega^2$$

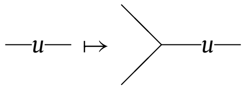
and that implies:

$$0 = t\sigma^p P_{gl} P_{osp}\left(-\frac{3}{2}\sigma\omega^2 + \frac{5}{2}t\sigma^2\omega - \frac{20}{9}t^3\sigma\omega\right) = -\frac{1}{18}t\sigma^{p+1}\omega P_{gl} P_{osp} P_{exc}.$$

Therefore $t\sigma^p\omega P_{gl} P_{osp} P_{exc}$ is zero in Λ for every $p \geq 0$ and that finishes the proof. $\square$

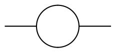
A particular consequence of this result is the fact that a cobracket morphism is not necessarily injective:

Proposition 8.5. *The morphism:*

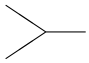


from $F(2)$ to $F(3)$ is not injective.

Proof. Denote this morphism by f. Let U be the image of $\omega P_{gl}P_{osp}P_{exc}$ under the morphism $\varphi : R_0 \longrightarrow \Lambda$. We have: $U \neq 0$ and $tU = 0$. Consider the following element in $F(2)$:

$u = U$ 

Because of Corollary 4.6 u is nonzero. But its image under f is:

$2tU$  $= 0$

and the result follows. □

Conjecture. *Let J be the ideal of R generated by $t\omega P_{gl}P_{osp}P_{exc}$. Then the morphism $\varphi : R_0 \longrightarrow \Lambda$ induces an isomorphism from R_0/J to Λ.*

Appendix: the Hamiltonian Lie superalgebra $\widehat{H}(n)$

This section is devoted to the construction of the Lie superalgebra $\widehat{H}(n)$ considered in the proof of Theorem 8.3.

Let $x_1, x_2, \ldots, x_n$ be formal variables (with $n > 0$). Let E be the exterior algebra on these variables. This algebra is graded by considering each x_i as a degree 1 variable. For each i there is a derivation ∂_i sending x_i to 1 and the other variables to 0. So we can define a bracket on E by:

$$[u, v] = \sum_i (-1)^{|u|}\partial_i(u)\wedge\partial_i(v)$$

where $|u|$ is the degree of u. Let f be the linear form on E of degree $-n$ sending $x_1\wedge x_2\wedge\cdots\wedge x_n$ to 1.

Proposition A.1. *Let $\widehat{H}(n)$ be the module E with the degree shifted by -2. Then the bracket $[\ ,\ ]$ induces on $\widehat{H}(n)$ a structure of Lie superalgebra. Moreover the form:*

$$u \otimes v \mapsto \langle u, v\rangle = f(u\wedge v)$$

is a nonsingular invariant supersymmetric form on $\widehat{H}(n)$ of degree $4 - n$.

The center of $\widehat{H}(n)$ is generated by 1. The derived algebra $[\widehat{H}(n), \widehat{H}(n)]$ is the kernel of f. Moreover the quotient of $\widehat{H}(n)$ by its center is isomorphic to the Hamiltonian Lie superalgebra $\widetilde{H}(n)$.

Proof. See [12] for a description of Hamiltonian algebras $H(n)$ and $\widetilde{H}(n)$. The morphism from $\widehat{H}(n)$ to $\widetilde{H}(n)$ is given by:

$$x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_p} \mapsto \sum_{1\le k\le p} (-1)^{k-1}\theta_{i_1}\theta_{i_2}\ldots\widehat{\theta_{i_k}}\ldots\theta_{i_p}\frac{\partial}{\partial\theta_{i_k}}$$

and the proposition is easy to check. □

Proposition A.2. *For $n = 1$ or n even the module $H^2(\widehat{H}(n))$ is trivial and $\widehat{H}(n)$ has no central extension. If n is odd and bigger then 2, $H^2(\widehat{H}(n))$ is 1-dimensional and generated by the cocycle $u \otimes v \mapsto f(u)f(v)$.*

Proof. Let φ be a 2-cocycle. In order to determine φ we will need some notations:

— A vector in E is called basic if it is a product of distinct x_i's (up to sign).

— The degree of a basic vector u is denoted by $|u|$.

— The support of a basic vector $e = \pm x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_p}$ is the set $\{x_{i_1}, \ldots, x_{i_p}\}$.

— $\mathcal{B}$ is the set of collections of basic vectors with disjoint supports.

So we have the following:

$$\forall (u, v, w) \in \mathcal{B}, \qquad [u\wedge v, u\wedge w] = \begin{cases} (-1)^{|u|+|v|} v\wedge w & \text{if } |u| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since φ is a 2-cocycle, the following condition

$$(*)\quad (-1)^{|u||w|}\varphi([u, v] \otimes w) + (-1)^{|v||u|}\varphi([v, w] \otimes u) + (-1)^{|w||v|}\varphi([w, u] \otimes v) = 0$$

holds for every basic vectors u, v, w.

Consider three basic vectors u, v, w. There exist $(e, \alpha, \beta, \gamma, x, y, z)$ in $\mathcal{B}$ such that:

$$u = e\wedge\beta\wedge\gamma\wedge x \qquad v = e\wedge\gamma\wedge\alpha\wedge y \qquad w = e\wedge\alpha\wedge\beta\wedge z$$

and the only possibilities for which $[u, v]$ or $[v, w]$ or $[w, u]$ is nonzero are the following (up to a cyclic permutation):

$$\begin{array}{llll}
|e| = 1, & |\alpha| = |\beta| = |\gamma| = 0 & & \\
|e| = 1, & |\alpha| = |\beta| = 0, & |\gamma| > 0 & \\
|e| = 1, & |\alpha| = 0, & |\beta| > 0, & |\gamma| > 0 \\
|e| = 0, & |\alpha| = 1, & |\beta| = |\gamma| = 0 & \\
|e| = 0, & |\alpha| = 1, & |\beta| > 1, & |\gamma| = 0 \\
|e| = 0, & |\alpha| = 1, & |\beta| > 1, & |\gamma| > 1 \\
|e| = 0, & |\alpha| = |\beta| = 1, & |\gamma| = 0 & \\
|e| = 0, & |\alpha| = |\beta| = 1, & |\gamma| > 1 & \\
|e| = 0, & |\alpha| = |\beta| = |\gamma| = 1. & &
\end{array}$$

By applying the condition $(*)$ to all these cases we get the following relations:

$(R1)$ $(-1)^{|x||z|}\varphi(x\wedge y \otimes z\wedge e) + (-1)^{|y||x|}\varphi(y\wedge z \otimes x\wedge e) + (-1)^{|z||y|}\varphi(z\wedge x \otimes y\wedge e) = 0$

$(R2)$ $\varphi(\gamma\wedge z\wedge y \otimes x\wedge e\wedge\gamma) = (-1)^{|x||y|+|\gamma|+1}\varphi(\gamma\wedge z\wedge x \otimes y\wedge e\wedge\gamma)$

$(R3)$ $\varphi(\beta\wedge\gamma\wedge y\wedge z \otimes \beta\wedge\gamma\wedge e\wedge x) = 0$

$(R4)$ $\varphi(y\wedge z \otimes x) = 0$

$(R5)$ $\varphi(\beta\wedge y\wedge z \otimes \beta\wedge x) = 0$

$(R6)$ $\varphi(\beta\wedge\gamma\wedge y\wedge z \otimes \beta\wedge\gamma\wedge x) = 0$

$(R7)$ $(-1)^{|x||y|+|x|+|y|}\varphi(\beta\wedge y\wedge z \otimes \beta\wedge x) = (-1)^{|y||z|}\varphi(\alpha\wedge z\wedge x \otimes \alpha\wedge y)$

$(R8)$ $(-1)^{|x||y|+(|\gamma|+1)(|x|+|y|)}\varphi(\beta\wedge\gamma\wedge y\wedge z \otimes \beta\wedge\gamma\wedge x) = (-1)^{|y||z|+|\gamma|}\varphi(\gamma\wedge\alpha\wedge z\wedge x \otimes \gamma\wedge\alpha\wedge y)$

$(R9)$ $(-1)^{|x||z|}\varphi(\alpha\wedge\beta\wedge x\wedge y \otimes \alpha\wedge\beta\wedge z) + (-1)^{|y||x|}\varphi(\beta\wedge\gamma\wedge y\wedge z \otimes \beta\wedge\gamma\wedge x) + (-1)^{|z||y|}\varphi(\gamma\wedge\alpha\wedge z\wedge x \otimes \gamma\wedge\alpha\wedge y) = 0.$

Using relations (R4) and (R1) with $|x| = |y| = 0$ and $|z| = n - 1$ we get:

$$\forall (u, v) \in \mathcal{B}, \quad \varphi(u \otimes v) = 0.$$

Using relation (R5) we get:

$$\forall (u, v, w) \in \mathcal{B}, \quad |u| > 1, |u| + |v| + |w| < n \Longrightarrow \varphi(u\wedge v \otimes u\wedge w) = 0.$$

With the relation (R3) we get:

$$\forall (u, v, w) \in \mathcal{B}, \quad 1 < |u| < n, |u| + |v| + |w| = n \Longrightarrow \varphi(u\wedge v \otimes u\wedge w) = 0.$$

The relation (R2) implies:

$$\forall (u, v, w) \in \mathcal{B}, \quad |u| = 1, |w| > 0 \Longrightarrow \varphi(u\wedge v \otimes w\wedge u) = \varphi(u \otimes v\wedge w\wedge u)$$

and since φ is antisymmetric:

$$\forall (u, v, w) \in \mathcal{B}, \quad |u| = 1 \Longrightarrow \varphi(u\wedge v \otimes w\wedge u) = \varphi(u \otimes v\wedge w\wedge u).$$

Finally the relation (R7) implies:

$$\forall (u, v, w) \in \mathcal{B}, \quad |u| = |v| = 1 \Longrightarrow \varphi(u \otimes w\wedge u) = \varphi(v \otimes w\wedge v)$$

and $\varphi(u \otimes w\wedge u)$ depends only on w (if $|u| = 1$) and $\varphi(u\wedge v \otimes w\wedge u)$ depends only on $[u\wedge v, w\wedge u]$. Therefore there exist a linear morphism g and a scalar c such that:

$$\varphi(u \otimes v) = g([u, v]) + cf(u)f(v)$$

for every u and v in $\widehat{H}(n)$.

On the other hand φ is antisymmetric and: $c(1 + (-1)^n) = 0$.

If n is even, $c = 0$ and φ is a coboundary. Then $H^2(\widehat{H}(n))$ is trivial.

If $n = 1$, $u \otimes v \mapsto f(u)f(v)$ is a coboundary and $H^2(\widehat{H}(n))$ is also trivial.

If $n > 2$ is odd, $H^2(\widehat{H}(n))$ is 1-dimensional and generated by the cocycle $u \otimes v \mapsto f(u)f(v)$. $\square$

Corollary A.3. *For $n > 1$, $H^2(H(n))$ is a 1-dimensional module generated by the central extension $[\widehat{H}(n), \widehat{H}(n)] \longrightarrow H(n)$.*

Proof. Let L be the algebra $[\widehat{H}(n), \widehat{H}(n)]$ and L_0 be the quotient $\widehat{H}(n)/L$. By looking in low degree the spectral sequence of the cohomology of the extension:

$$0 \longrightarrow L \longrightarrow \widehat{H}(n) \longrightarrow L_0 \longrightarrow 0$$

we get the following:

$$n \text{ even} \implies H^1(L) \simeq H^2(L) \simeq 0$$

$$n = 1 \implies d_2 : H^1(L) \xrightarrow{\simeq} H^2(L_0)$$

$$n > 2,\ n \text{ odd} \implies \mathrm{H}^1(L) \simeq 0 \quad \text{and} \quad d_3 : H^2(L) \longrightarrow H^3(L_0) \text{ is injective.}$$

So for $n > 1$, $H^1(L)$ is trivial.

Let Z be the center of L. The spectral sequence of the central extension:

$$0 \longrightarrow Z \longrightarrow L \longrightarrow H(n) \longrightarrow 0$$

implies that $H^1(H(n))$ is trivial and the morphism d_2 is an isomorphism from $H^1(Z)$ to $H^2(H(n))$. □

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